

PARTIAL COMPACTIFICATION OF MONOPOLES AND METRIC ASYMPTOTICS

CHRIS KOTTKE AND MICHAEL SINGER

ABSTRACT. We construct a partial compactification of the moduli space, \mathcal{M}_k , of $SU(2)$ magnetic monopoles on \mathbb{R}^3 , wherein monopoles of charge k decompose into widely separated ‘monopole clusters’ of lower charge going off to infinity at comparable rates. The hyperKähler metric on \mathcal{M}_k has a complete asymptotic expansion up to the boundary, the leading term of which generalizes the asymptotic metric discovered by Bielawski, Gibbons and Manton in the case that each lower charge is 1.

CONTENTS

1. Introduction	2
1.1. Overview of the construction	6
1.2. Outlook: monopole compactification	9
Acknowledgements	9
2. The Bogomolny equations on a scattering manifold	10
2.1. Gauge theory on a manifold with boundary	10
2.2. \mathbb{R}^3 as a scattering manifold	12
2.3. Bogomolny equations	13
2.4. The boundary connection and symmetry-breaking	15
2.5. Framed moduli space	17
2.6. The local structure of \mathcal{M}_k	18
2.7. The centre of a monopole	20
2.8. Structure of the tangent space	21
2.9. Dirac monopoles	23
3. Formal 1-parameter families	27
3.1. Gluing space	27
3.2. Metric structure	28
3.3. Bogomolny equation	32
3.4. Ideal monopoles and pregluing configurations	34
3.5. The iteration	37
3.6. Polyhomogeneity	38
3.7. Formal solution	40
4. Moduli of ideal monopoles	45
4.1. Moduli of ideal monopoles for a fixed configuration of points	45
4.2. Configurations of points	46
4.3. Moduli of ideal monopoles	47

5. Universal gluing space and parameterized gluing	54
5.1. Bogomolny equation on \mathcal{Z}	55
5.2. Differential and normal operators	57
5.3. Sobolev spaces	59
5.4. Gauge fixing	61
5.5. True solution	63
6. The metric	67
6.1. Infinitesimal translations	70
Appendix A. Sobolev spaces	71
A.1. Sobolev spaces for X_i	73
Appendix B. Coulomb gauge	73
Appendix C. Linear analysis	75
C.1. Linear analysis of L_{X_i}	75
C.2. Linear analysis of L_D	83
C.3. Linear analysis for Coulomb gauge	85
Appendix D. Pseudodifferential operators	88
D.1. The double spaces	89
D.2. The triple spaces	97
D.3. Pseudodifferential operators	100
D.4. Sobolev spaces and mapping properties	104
D.5. Residual ideals	108
D.6. Parametrices	109
References	112

1. INTRODUCTION

This paper is the first in a series aimed at studying the asymptotic regions of the monopole moduli spaces and the behaviour of the L^2 metric in these regions.

Recall that an $SU(2)$ monopole is a (gauge equivalence class) of solutions of the *Bogomolny equations*

$$*F_A = \nabla_A \Phi, \quad (1.1)$$

where (A, Φ) is a pair consisting of a connection A on a principal $SU(2)$ -bundle P over \mathbb{R}^3 and Φ , the *Higgs field*, is a section of the associated adjoint bundle $\text{ad}(P)$. The Bogomolny equations are supplemented by assuming that the *Yang–Mills–Higgs* action is finite,

$$\int_{\mathbb{R}^3} (|F_A|^2 + |\nabla_A \Phi|^2) < \infty, \quad (1.2)$$

and that

$$|\Phi(z)| \rightarrow 1 \text{ as } |z| \rightarrow \infty \text{ in } \mathbb{R}^3, \quad (1.3)$$

see below for further discussion. Then (1.3) entails that the degree of Φ over a large sphere in \mathbb{R}^3 is a positive integer k , the *magnetic charge* (or monopole

number). By the device of framing (A, Φ) at infinity, and restricting to the gauge group \mathfrak{G}_0 of elements of $\text{Aut}(P)$ that approach the identity at infinity, the *moduli space* \mathcal{M}_k of *framed monopoles of charge k* is defined as the set of solutions of (1.1) satisfying (1.2) and (1.3), divided by the action of \mathcal{G}_0 . It is known that \mathcal{M}_k is a smooth manifold of real dimension $4k$, non-compact, but carrying a complete riemannian L^2 metric G_k [AH88, Tau83, Tau85]. More precisely, if (A, Φ) represents a point m of \mathcal{M}_k , then

$$T_m \mathcal{M}_k = L^2\text{-Ker}(L_{A,\Phi}) \quad (1.4)$$

where $L_{A,\Phi}$ is the linear operator

$$L_{A,\Phi} : C^\infty(\mathbb{R}^3, \Lambda \otimes \text{ad}(P)) \longrightarrow C^\infty(\mathbb{R}^3, \Lambda \otimes \text{ad}(P)), \quad (1.5)$$

and

$$L_{A,\Phi} = \begin{bmatrix} *d_A & -d_A \\ -d_A^* & 0 \end{bmatrix} + \text{ad}(\Phi) \otimes \text{Id} \quad (1.6)$$

and $\Lambda = \Lambda^1 \oplus \Lambda^0$. The ‘top row’ of this operator is the linearization at (A, Φ) of the Bogomolny equations; the bottom row is the Coulomb gauge fixing condition that (a, ϕ) is L^2 -orthogonal to the tangent space of the gauge orbit containing (A, Φ) . The Riemannian metric, G_k , on the moduli space is defined by the formula

$$\|(a, \phi)\|_{G_k}^2 = \int_{\mathbb{R}^3} |a|^2 + |\phi|^2 \quad (1.7)$$

for (a, ϕ) in the tangent space (1.4). Here we have chosen the $\text{SU}(2)$ -invariant metric $-\frac{1}{2} \text{tr}(A^2)$ on the Lie algebra $\mathfrak{su}(2)$.

The metric G_k is hyperKähler [HKLR87, AH88]. It is known that G_1 is the flat metric on $\mathbb{R}^3 \times S^1$ and that G_2 is essentially the riemannian product of $\mathbb{R}^3 \times S^1$ and the famous Atiyah–Hitchin metric [AH88]. For $k \geq 3$ it is not feasible to compute G_k explicitly but one may hope to gain a partial understanding of it asymptotically in terms of the metrics G_{k_j} on moduli spaces of lower charge.

In order to explain this idea more carefully, recall [AH88, Prop. 3.8] about the asymptotic structure of \mathcal{M}_k . The following statement uses the fact that a monopole $m \in \mathcal{M}_k$ has a well-defined centre of mass (cf. §2.7); denote by \mathcal{M}_k^c the moduli space of monopoles centred at the origin, a submanifold of \mathcal{M}_k of dimension $4k - 3$.

Theorem 1.1 ([AH88], Proposition 3.8¹). *Given an infinite sequence of points of \mathcal{M}_k , there exists a subsequence (m_ν) , a partition $k = k_0 + \dots + k_N$ with $k_i > 0$, $i \geq 1$ (we allow $k_0 = 0$) and sequences of points $(z_\nu^i) \in \mathbb{R}^3$ ($i = 0, \dots, N$) such that*

- (i) *the sequence $m_\nu(\cdot - z_\nu^i)$ converges weakly (i.e. on compact subsets of \mathbb{R}^3) to a k_i -monopole m^i , centered at the origin,*

¹We have slightly rephrased the statement of this Proposition.

(ii) as $\nu \rightarrow \infty$, the $|z_\nu^i z_\nu^j| \rightarrow \infty$ for $i \neq j$ and the unit vectors

$$\frac{\overrightarrow{z_\nu^i z_\nu^j}}{|z_\nu^i z_\nu^j|}$$

converge in \mathbb{S}^2 . Here we assume that $z_\nu^0 \rightarrow z^0$ converges in \mathbb{R}^3 , so $|z_\nu^i| \rightarrow \infty$ for $i \geq 1$.

In fact, Taubes [Tau85] has proved more refined results, showing that, along appropriate sequences, \mathbb{R}^3 can be divided into ‘strong-field’ regions in which most the energy (1.2) is concentrated, the centers of whose path components can be associated with the sequences (z_ν^i) above, and a ‘weak-field’ region in which the fields are approximately abelian, and to high order satisfy the abelian, or ‘Dirac’, monopole equations (cf. §2.9 below), with prescribed singularities approaching the strong-field regions. (This dichotomy is reflected in our geometric construction below.)

Thus the non-compactness of \mathcal{M}_k is captured by sequences of monopoles of lower charge escaping to ∞ in \mathbb{R}^3 . Note that in the above theorem, there is no control on the relative sizes of the $|z_\nu^i z_\nu^j|$ for different pairs ij . The simplest part of the asymptotic region of \mathcal{M}_k , the subject of this paper, corresponds to the case that all these lengths are uniformly comparable as $\nu \rightarrow \infty$.

The asymptotic behaviour of G_k has been studied in special cases by various authors. In [GM95], Gibbons and Manton derived a model metric for the asymptotic region corresponding to $k_0 = 0$ and $k_i = 1$, $i \geq 1$, and in [Bie98] Bielawski proved that the L^2 metric is exponentially close to the model in this region. Using the representation of monopoles via spectral curves, Bielawski also proved the existence of simplified hyperKähler models for cluster regions of higher charge in [Bie08], though a description of these models in terms of the metrics on lower moduli spaces appears to be difficult to obtain directly from this work.

To study these asymptotic regions directly, we construct a partial compactification of \mathcal{M}_k by associating boundary hypersurfaces to these limits. These boundary hypersurfaces are moduli spaces of *ideal monopoles*, objects which roughly speaking consist of the following data (they are defined more precisely below): a list $\underline{k} = (k_0, k_1, \dots, k_N)$, of integers with $k_0 \geq 0$, $k_i \geq 1$ for $i = 1, \dots, N$, and $\sum_{i=0}^N k_i = k$; a collection of monopoles respectively of charges k_0, \dots, k_N ; and an asymptotic configuration of distinct non-zero points $(\zeta_1, \dots, \zeta_N)$ up to scale. To be more specific, define

$$\mathcal{E}_N^* = \left\{ \underline{\zeta} = (\zeta_1, \dots, \zeta_N) \in (\mathbb{R}^3)^N : \zeta_i \neq \zeta_j, \text{ for } i \neq j, \zeta_i \neq 0, \sum_i |\zeta_i|^2 = 1 \right\} \quad (1.8)$$

which we view as part of the boundary of the radial compactification of \mathbb{R}^{3N} . The moduli space, $\mathcal{I}_{\underline{k}}$, of ideal monopoles of type \underline{k} is then a non-trivial fibre

bundle

$$\mathcal{I}_{\underline{k}} \longrightarrow \mathcal{E}_N^* \quad (1.9)$$

over the space of ideal configurations \mathcal{E}_N^* , with fibre

$$(\mathcal{I}_{\underline{k}})_{\underline{\zeta}} = \mathcal{M}_{k_0} \times \mathcal{M}_{k_1}^c \times \cdots \times \mathcal{M}_{k_N}^c. \quad (1.10)$$

The $\mathcal{I}_{\underline{k}}$ (or more correctly, their quotients by a symmetric group interchanging factors of equal charge) form the boundary hypersurfaces of our partial compactification, with normal coordinate given by a scaling parameter ε , defined so that the $z^i = \zeta_i/\varepsilon$. Thus the parameters in the base, \mathcal{E}_N^* , represent the limiting locations of the monopole clusters of charges k_i , $i = 1, \dots, N$ which have gone off to infinity, with a cluster of charge k_0 which remains behind, while the parameters in the fiber represent the (possibly recentered) clusters themselves.

In §4, we shall describe the structure of the bundle $\mathcal{I}_{\underline{k}}$ over \mathcal{E}_N^* . The type \underline{k} determines a rank $N+1$ *Gibbons-Manton torus bundle* $\mathcal{T}_{\text{GM}} \longrightarrow \mathcal{E}_N^*$ with respect to which $\mathcal{I}_{\underline{k}}$ is the associated fiber bundle

$$\mathcal{I}_{\underline{k}} = \mathcal{T}_{\text{GM}} \times_{\text{U}(1)^{N+1}} (\mathcal{M}_{k_0} \times \mathcal{M}_{k_1}^c \times \cdots \times \mathcal{M}_{k_N}^c) \longrightarrow \mathcal{E}_N^*$$

given by the quotient by $\text{U}(1)^{N+1}$ acting on \mathcal{T}_{GM} on the left and by the circle actions on the framed moduli spaces on the right. This description generalizes the case studied by Gibbons–Manton and Bielawski, where $k_0 = 0$, $k_i = 1$, the centred moduli spaces are reduced to circles, and so the asymptotic moduli space is just the Gibbons–Manton torus bundle itself of type $(0, 1, \dots, 1)$ over \mathcal{E}_k^* .

Our first main result is

Theorem 1.2. *For each $\iota_0 \in \mathcal{I}_{\underline{k}}$, there exists a neighborhood $\mathcal{U} \ni \iota_0$, $\varepsilon_0 > 0$, and a smooth map*

$$\Psi : \mathcal{U} \times (0, \varepsilon_0) \longrightarrow \mathcal{M}_k \quad (1.11)$$

which is a local diffeomorphism onto its image, and such that $\Psi(\iota, \varepsilon) \longrightarrow \iota$ as $\varepsilon \longrightarrow 0$, with a complete asymptotic expansion in ε .

The map itself depends on the choice of gauge representative for ι_0 , though two choices will agree to leading order in ε .

Monopole gluing theorems are not new; indeed, Taubes’ gluing theorem in [JT80] for widely separated charge 1 monopoles was the first existence result for monopoles of higher charge. More recently, Oliveira [Oli14] and Foscolo [Fos14] have obtained gluing results for monopoles on asymptotically conic 3-manifolds, and on $\mathbb{R}^2 \times \mathbb{S}^1$, respectively. In Donaldson’s representation [Don84] of \mathcal{M}_k as a space of rational maps, gluing corresponds simply to addition of rational maps, though metric information is not readily available in this picture. The chief advantage of our approach over more traditional techniques is that we obtain complete asymptotic expansions in the gluing parameter, and in particular can compute the metric to leading order in this parameter.

The tangent bundle of $\mathcal{U} \times (0, \varepsilon_0)$ can be identified with the product

$$T\mathcal{M}_{k_0} \oplus \left(\bigoplus_{i=1}^N T\mathcal{M}_{k_i}^c \oplus \mathbb{R}^3 \right). \quad (1.12)$$

Here the N factors of \mathbb{R}^3 come from identifying $\mathcal{E}_N^* \times (0, \varepsilon_0)$ with the configuration space \mathcal{C}_N^* of distinct non-zero points, *not* up to scaling, which is an open subset of $(\mathbb{R}^3)^N$. Our second main result states

Theorem 1.3. *The pulled back metric $\Psi^*(G_k)$ has a complete asymptotic expansion as $\varepsilon \rightarrow 0$, with leading order*

$$\Psi^*(G_k) \cong G_{k_0} \oplus \left(\bigoplus_{i=1}^N G_{k_i}^c \oplus 2\pi k_i \eta_i \right) + \mathcal{O}(\varepsilon) \quad (1.13)$$

with respect to the identification with (1.12).

Further refinements (not proved here) give the next order term in the metric as well, with the result that $\Psi^*(G_k)$ generalizes the asymptotic metric of Gibbons and Manton [GM95] for the case where $k_0 = 0$, $k_i = 1$, $i = 1, \dots, N$.

Remark. The structure of this compactification, and in particular the Gibbons–Manton bundles associated to a general \underline{k} , is also known to Bielawski. More details will appear in [Bie].

1.1. Overview of the construction. We give a brief overview of our construction, highlighting the advantages of our approach, and explaining how ideal monopoles enter.

The first step is to pass to the radial compactification, $X = \overline{\mathbb{R}^3}$, of \mathbb{R}^3 as a convenient way to deal with the non-compactness of \mathbb{R}^3 . Having done so, the Euclidean metric becomes a smooth, bounded, positive definite metric on the so-called scattering tangent bundle ${}^{\text{sc}}TX$ of X , and monopoles may be defined in terms of data which are smooth up to ∂X and are regarded as sections of $\bigwedge^j {}^{\text{sc}}T^*X \otimes \mathfrak{p}$, where we introduce the notation $\mathfrak{p} = \text{ad}(P)_{\mathbb{C}}$. This point of view, as applied to the classical theory of monopoles on \mathbb{R}^3 , is summarized in §2.

The next step is to incorporate the parameter ε into the geometry of the problem. We begin with the product $Z_0 = X \times [0, \infty)_{\varepsilon}$, equipping each fiber of the projection $\varrho : Z_0 \rightarrow [0, \infty)$ with the Euclidean metric on X . Fixing a configuration $\underline{\zeta} \in \mathcal{E}_N^*$, the paths $(z(\varepsilon), \varepsilon) = (\zeta_j/\varepsilon, \varepsilon)$ approach the corner $\partial X \times \{0\} \subset Z_0$, and we let

$$Z_1 = [Z_0; \partial X \times \{0\}]$$

be the real blow-up of this corner in Z_0 . The new boundary face obtained by this blow-up, denoted by D , is diffeomorphic to a cylinder, with a natural interpretation as the blow-up $D \cong [\overline{\mathbb{R}^3}; \{0\}]$ of the radial compactification of \mathbb{R}^3 at the origin. Thus equipped with the interior Euclidean coordinate

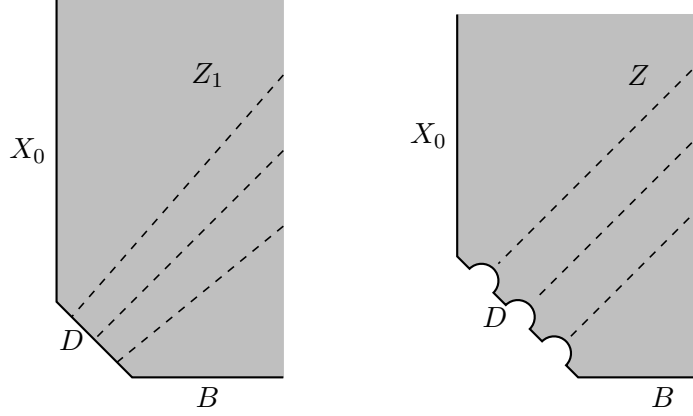


FIGURE 1. The space Z_1 with the lifts of the curves $z = \zeta_j/\varepsilon$ hitting the new face D transversely in points ζ_j , and the blow-up of this to the space Z .

ζ , D meets the boundaries of the lifted curves $z(\varepsilon) = \zeta_i/\varepsilon$ transversally at the points where $\zeta = \zeta_j$. (See Figure 1.)

The geometric construction is completed by blowing these points up in D , setting

$$Z = [Z_1; \{\zeta_1, \dots, \zeta_N\}].$$

The new faces are denoted by X_i , $i = 1, \dots, N$, and the lift of the original face $X \times \{0\}$ is denoted by X_0 (see Figure 1). These admit canonical isometries $X_i \cong X$ with the radial compactification of \mathbb{R}^3 with respect to the lift of the original fiberwise Euclidean metric on Z_0 .

We also lift the projection to the parameter interval to a map $\varrho : Z \rightarrow [0, \infty)_\varepsilon$. For $\varepsilon > 0$, the fiber $\varrho^{-1}(\varepsilon)$ is canonically isometric to the original radially compactified \mathbb{R}^3 , whereas $\varrho^{-1}(0)$ is a manifold ‘with normal crossings’ consisting of (the lifted) $D \cong [\overline{\mathbb{R}^3}; \{0, \zeta_1, \dots, \zeta_N\}]$ and the X_i , identified along their boundaries. We denote by ρ_D , ρ_X , and ρ_B choices of respective boundary defining functions for D , $X_0 \cup X_1 \cup \dots \cup X_N$, and B , the lift of the original boundary face $\partial X \times [0, \infty)$ of Z_0 . We may assume that $\varepsilon = \rho_D \rho_X$.

Next, we fix a principal $SU(2)$ bundle P over Z and consider the fiberwise Bogomolny operator

$$\mathcal{B}(A, \Phi) = *_\varrho F_{A_\varrho} - \nabla_{A_\varrho} \Phi_\varrho$$

on pairs, (A, Φ) , of connection and Higgs bundle on P , where the notation indicates that the data are restricted to each fiber $\varrho^{-1}(\varepsilon)$ before computing the curvature, covariant derivative, and Hodge star operator.

One checks that if (A, Φ) is smooth up to the boundaries of Z , then, as a section of the appropriate fiberwise 1-form bundle twisted by \mathfrak{p} , $\mathcal{B}(A, \Phi)$ is smooth, and in fact

$$\mathcal{B}(A, \Phi) = \mathcal{O}(\rho_D). \quad (1.14)$$

We ultimately embark on the process of improving the error term on the right hand side of (1.14) until it vanishes identically for small ε , thereby obtaining 1-parameter families of monopoles on the fibers $\varrho^{-1}(\varepsilon)$ for $\varepsilon > 0$.

The restriction of \mathcal{B} to X_i is nothing other than the Euclidean Bogomolny equation $*F_{A|X_i} - \nabla_{A|X_i}(\Phi|X_i)$, so if we choose (A, Φ) so that its restriction, $(A, \Phi)|_{X_i}$, solves this equation for each $i = 0, \dots, N$, then (1.14) improves by a factor of ρ_X and

$$\mathcal{B}(A, \Phi) = \mathcal{O}(\rho_D \rho_X) = \mathcal{O}(\varepsilon). \quad (1.15)$$

Over D , the leading part of the Bogomolny operator is just $\nabla_{A|D}(\Phi|D)$, so choosing this to vanish improves the approximation by a factor of ρ_D . The sub-leading order part of the Bogomolny operator over D imposes a condition on $A|D$. In more detail, $A|D$ reduces to a connection on a $U(1)$ -bundle over D determined by $\Phi|D$, and must solve the $U(1)$ monopole equations there, with prescribed conic singularities at $\{\zeta = \zeta_j : j = 0, \dots, N\}$. Imposing this condition improves the error term to

$$\mathcal{B}(A, \Phi) = \mathcal{O}(\rho_D^3 \rho_X) = \mathcal{O}(\varepsilon \rho_D^2). \quad (1.16)$$

An ideal monopole is properly defined to be the restriction to $\varrho^{-1}(0) = D \cup X_0 \cup \dots \cup X_N$ of a smooth configuration (A, Φ) on Z such that (1.16) holds. Ideal monopoles are acted on by a gauge group given by the restriction to $\varrho^{-1}(0)$ of gauge transformations on $P \rightarrow Z$ which are the identity at B . In fact, though there appears to be additional information represented by the Dirac monopole on the ‘interstitial region’, D , where the symmetry is broken down to $U(1)$, this extra information disappears when we consider gauge equivalence classes of ideal monopoles.

The choice of data (A, Φ) on Z , representing a given ideal monopole on $\varrho^{-1}(0)$, is the starting point for an iteration, carried out in §3, in which we successively improve the error term on the right hand side of (1.16). The output of this iteration is a section

$$(a, \phi) \in \mathcal{A}(Z; (\Lambda^1 \oplus \Lambda^0) \otimes \mathfrak{p})$$

with the property that

$$\mathcal{B}(A + a, \Phi + \phi) = \mathcal{O}(\rho_D^\infty \rho_X^\infty \rho_B^\infty).$$

Here the notation \mathcal{A} means that (a, ϕ) is smooth in the interior of Z and has complete asymptotic expansions in powers $\rho^j (\log \rho)^k$ for each of the boundary defining functions ρ , with compatibility between these expansions at the corners. (In precise terms, (a, ϕ) is *polyhomogeneous conormal*, c.f. [Mel92] and §3.6.)

The final step—which is carried out in §5 smoothly in families, on a space fibering over \mathcal{I} with fibers given by the Z —is a further correction $(\tilde{a}, \tilde{\phi}) \in \rho^\infty C^\infty(Z; (\Lambda^1 \oplus \Lambda^0) \otimes \mathfrak{p})$ such that

$$\mathcal{B}(A + a + \tilde{a}, \Phi + \phi + \tilde{\phi}) = 0 \text{ on } \varrho^{-1}([0, \varepsilon_0))$$

for some $\varepsilon_0 > 0$. This step follows from the construction of a pseudodifferential parametrix for the linearization of \mathcal{B} and an implicit function theorem argument.

Having constructed such smooth families, we then consider variations with respect to the parameters in §6. These represent solutions to the linearized, fiberwise Bogomolny equations on Z , which by small perturbation can be put into Coulomb gauge along each fiber. This gives the differential of the gluing map (1.11), and allows the metric $\Psi^*(G_k)$ to be computed by pairing two such variations and integrating over the fibers of $\varrho : Z \rightarrow [0, \varepsilon_0]$. The leading order contribution in ε , (1.13), reduces to explicit L^2 pairings of Euclidean monopole variations over the boundary faces X_i , the computations of which already appear in §2.

1.2. Outlook: monopole compactification. In future work with Melrose, we shall complete the compactification of \mathcal{M}_k as a smooth manifold with corners and study the metric on this compactification. (An alternative approach is being pursued by Bielawski [Bie].) We explain why a compactification as a manifold with corners is natural for this problem, illustrating with the simplest possible cases.

Let $k = 3$. According to our gluing theorem there are two asymptotic regions of \mathcal{M}_3 , let's call them \mathcal{V}_{111} and \mathcal{V}_{21} corresponding respectively to the partition $3 = 1 + 1 + 1$ and $3 = 2 + 1$. There is, however, a ‘transition region’ between \mathcal{V}_{111} and \mathcal{V}_{21} which we can describe in terms of log-smooth 2-parameter families $m(\varepsilon_1, \varepsilon_2) \in \mathcal{M}_3$ with the following properties. For fixed $\varepsilon_1 > 0$, $\varepsilon_2 \mapsto m(\varepsilon_1, \varepsilon_2)$ is a smooth curve in \mathcal{V}_{111} , while for $\varepsilon_2 > 0$, $\varepsilon_1 \mapsto m(\varepsilon_1, \varepsilon_2)$ is a smooth curve in \mathcal{V}_{21} . The parameter ε_2^{-1} is a measure of the separation of the charge-2 monopole in \mathcal{M}_2^c , so for smaller values ε_2 , this charge-2 monopole is going to infinity in \mathcal{M}_2^c . From the other point of view, on the curve $\varepsilon_1 \mapsto m(\varepsilon_1, \varepsilon_2)$, ε_2 is a measure of the distance between z_1 and z_2 relative to z_3 . Thus for small ε_2 , the configuration is becoming less and less widely separated, and the charge-1 monopoles centred at z_1 and z_2 have to be treated as a monopole of charge 2.

The construction of such smooth 2-parameter families will be the work of our next paper. Implemented systematically, this will allow us to build up a compactification of \mathcal{M}_k as a smooth manifold with corners, with control of the L^2 metric near each corner. The two-parameter families described in the previous paragraph correspond to codimension-2 corners of the moduli space.

Acknowledgements. The authors would like to thank Richard Melrose for his input into the approach we take to prove our main results. They also would like to thank Roger Bielawski, Nicholas Manton and Karsten Fritzsche for enlightening discussions.

2. THE BOGOMOLNY EQUATIONS ON A SCATTERING MANIFOLD

In this section we shall introduce smooth definitions of the monopole moduli spaces. Our approach is to pass to the radial compactification $X = \overline{\mathbb{R}^3}$ of \mathbb{R}^3 and regard the euclidean metric as a scattering metric on this space; we shall recall these notions below; but X is a compact manifold with boundary ∂X diffeomorphic to the 2-sphere \mathbb{S}^2 , and instead of the decay conditions when (A, Φ) are regarded as fields on \mathbb{R}^3 , we assume smoothness up to the boundary. The whole classical theory of monopoles can be developed in this setting and is equivalent to the ‘usual’ approach. The equivalence follows from the basic regularity theorems proved in Jaffe–Taubes [JT80] which show that the finite-energy conditions (1.2) imply decay properties of the fields equivalent to smoothness up to the boundary of X .

2.1. Gauge theory on a manifold with boundary. Let X be a manifold with boundary ∂X . Recall that a *boundary defining function* is a smooth function $\rho : X \rightarrow [0, \infty)$, such that $\partial X = \rho^{-1}(0)$ and $d\rho \neq 0$ on ∂X .

If p is any point of ∂X , then there exist adapted local coordinates (x, y) defined in a neighbourhood U of p in X such that $x \geq 0$, y is a system of local coordinates near p in ∂X , and such that $\rho|_U = x$.

Let $P \rightarrow X$ be a smooth principal G -bundle, where G is some Lie group. From here on, ‘smooth’ will always mean ‘smooth up to and including ∂X ’ unless otherwise stated, and the vector space of all smooth functions will be denoted by $C^\infty(X)$. It may be convenient to recall that we can also think of $C^\infty(X)$ as the space of restrictions of smooth functions from a slightly larger manifold \hat{X} in which X sits as the closed subset $\{\rho \geq 0\}$.

Definition 2.1. The space of smooth G -connections on P will be denoted by $\mathfrak{A}(X; P)$, or just \mathfrak{A} when there is no risk of confusion. The gauge group $\mathfrak{G}(X, P) = \mathfrak{G}$ is the space of smooth sections of $\text{Ad}(P)$, the bundle of groups associated to P . If $A \in \mathfrak{A}$, the curvature is denoted by F_A or $F(A)$ and is a smooth section of $\Lambda^2 T^*X \otimes \text{ad}(P)$.

Similarly we define the space of smooth Yang-Mills-Higgs (YMH) configurations:

Definition 2.2. Let P and X be as above. The space of smooth YMH (or monopole) configurations, $\mathfrak{C}(X, P)$ is defined to be

$$\mathfrak{C}(X, P) = \mathfrak{A}(X, P) \times C^\infty(X, \text{ad}(P)). \quad (2.1)$$

This will be abbreviated to $\mathfrak{C}(X)$ or even \mathfrak{C} when there is no risk of confusion.

The gauge group \mathfrak{G} acts on \mathfrak{C} by conjugation:

$$\gamma(A, \Phi) = (\gamma A, \text{Ad}(\gamma)\Phi) \quad (2.2)$$

and we have

$$F(\gamma(A)) = \text{Ad}(\gamma)F(A), \quad \nabla_{\gamma(A)}(\text{Ad}(\gamma)\Phi) = \text{Ad}(\gamma)\nabla_A\Phi, \quad (2.3)$$

where ∇_A is the covariant derivative operator

$$\nabla_A : C^\infty(X, \text{ad}(P)) \longrightarrow C^\infty(X, T^*X \otimes \text{ad}(P)). \quad (2.4)$$

Remark. We pause to emphasise what smoothness at the boundary means here. From the point of view of principal bundles, where connections are defined as smooth families of horizontal subspaces of TP , we mean that this family is smooth up to the boundary, and in particular admits a smooth extension as a connection on a principal bundle \hat{P} over the larger manifold \hat{X} . Alternatively, for the first-order differential operator (2.4), smoothness means that all coefficients are smooth when expressed in terms of smooth (up-to-the-boundary) local trivializations of the bundles.

2.1.1. Restriction and extension. Let $E \longrightarrow X$ be any smooth vector bundle, and let ∇_A be a smooth connection, identified with a covariant derivative operator, on E . Denote by $j : \partial X \longrightarrow X$ the inclusion. Denote by $E_\partial = j^*(E)$ the restriction of E to the boundary. As is well known we have a natural exact sequence a natural restriction map j^*

$$0 \longrightarrow \rho C^\infty(X, E) \longrightarrow C^\infty(X, E) \xrightarrow{j^*} C^\infty(\partial X, E_\partial) \longrightarrow 0. \quad (2.5)$$

We note the corresponding result for connections:

Proposition 2.3. *Let the data be above. Then there is a natural exact sequence*

$$0 \longrightarrow \rho C^\infty(X, {}^bT^*X \otimes \text{ad}(P)) \longrightarrow \mathfrak{A}(X, P) \xrightarrow{j^*} \mathfrak{A}(\partial X, P_\partial) \longrightarrow 0 \quad (2.6)$$

Here there is a harmless abuse of notation in that the second and third spaces are affine spaces, not vector spaces; we read from the sequence in particular that given any connection A_∂ on $P_\partial \longrightarrow \partial X$, there is a connection A on P with $j^*A = A_\partial$, and that A is unique up to the addition of an element in

$$\rho C^\infty(X, {}^bT^*X \otimes \text{ad}(P)). \quad (2.7)$$

Proof. This is reduced to the exactness of (2.5) by picking a background connection. In more detail, pick any connection A on P and put $j^*A = A_\partial$. To show that j^* is surjective, suppose B is any other connection on P_∂ . Then $B - A_\partial$ is a section of $C^\infty(\partial X, T^*Y \otimes \text{ad}(P_\partial))$. Now we use the variant

$$0 \longrightarrow \rho C^\infty(X, {}^bT^*X) \longrightarrow C^\infty(X, T^*X) \longrightarrow C^\infty(\partial X, T^*\partial X) \longrightarrow 0 \quad (2.8)$$

of (2.5) to find an extension \tilde{a} of $B - A_\partial$ to X . (See §2.2.1 below for the b-cotangent bundle ${}^bT^*$.)

Then $A + \tilde{a}$ is an extension of A_∂ to X . Using the exactness of (2.8) we obtain similarly that any two extensions of A_∂ differ by an element of (2.7).

Naturality corresponds to the statement that if $A_\partial = j^*(A)$, then

$$\nabla_{A_\partial} j^*(s) = j^*(\nabla_A s). \quad (2.9)$$

□

The restriction to the boundary A_∂ of A is then a connection on $E|_{\partial X}$, and induces a covariant derivative operator

$$\nabla_{A_\partial} : C^\infty(\partial X, E|_{\partial X}) \longrightarrow C^\infty(\partial X, T^*\partial X \otimes (E|_{\partial X})). \quad (2.10)$$

2.2. \mathbb{R}^3 as a scattering manifold. In order to write down the Bogomolny equations in this setting, we replace \mathbb{R}^3 by its radial compactification X [Mel94]. This can be defined as follows: first identify $\mathbb{R}^3 \setminus 0$ with $(0, \infty)_r \times \mathbb{S}^2$, where r is distance from 0. Now glue the half-closed cylinder $[0, \infty)_x \times \mathbb{S}^2$ to $\mathbb{R}^3 \setminus 0$ by the identification $x = r^{-1}$ to obtain the radial compactification X . The boundary $x = 0$ corresponds to the ‘sphere at infinity’ in \mathbb{R}^3 . Equip X with the C^∞ structure so that x is a (local) boundary defining function.

If y stands for local coordinates on the boundary \mathbb{S}^2 , the euclidean metric takes the form

$$\eta = \frac{dx^2}{x^4} + \frac{h(x, y, dy)}{x^2} \quad (2.11)$$

near $x = 0$, where for each x , $h(x, y, dy)$ represents a metric on \mathbb{S}^2 such that $h(0, y, dy)$ is the round metric. In particular it does not extend smoothly as a metric on TX . There is, however, a replacement, the ‘scattering tangent bundle’ ${}^{\text{sc}}TX$ which we now recall.

Denote by $\mathcal{V}(X)$ the space of all smooth vector fields on X and by $\mathcal{V}_{\text{sc}}(X)$ the subspace of those of finite length with respect to η . As shown in [Mel94], the subspace $\mathcal{V}_{\text{sc}}(X)$ is the full space of C^∞ sections of the *scattering tangent bundle* ${}^{\text{sc}}TX \longrightarrow X$, equipped with a smooth bundle map

$$\iota : {}^{\text{sc}}TX \longrightarrow TX \quad (2.12)$$

which is an isomorphism over the interior of X . More precisely, the map induced by ι on global sections

$$\iota : C^\infty(X, {}^{\text{sc}}TX) \longrightarrow C^\infty(X, TX) \quad (2.13)$$

gives an isomorphism $C^\infty(X, {}^{\text{sc}}TX) = \mathcal{V}_{\text{sc}}(X)$.

If p is a point of ∂X and we choose adapted local coordinates (x, y_1, y_2) in a neighbourhood U of p , then $\mathcal{V}_{\text{sc}}(X)$ is locally spanned by the elements

$$x^2 \partial_x, x \partial_{y_1}, x \partial_{y_2}. \quad (2.14)$$

and there is a basis e_0, e_1, e_2 of ${}^{\text{sc}}TU$ such that

$$\iota(e_0) = x^2 \partial_x, \iota(e_1) = x \partial_{y_1}, \iota(e_2) = x \partial_{y_2}. \quad (2.15)$$

We shall often abuse notation by regarding the vector fields (2.14) as a local frame for ${}^{\text{sc}}TX$.

We note that (tautologically) the euclidean metric η extends from \mathring{X} to define a smooth metric on ${}^{\text{sc}}TX$; equivalently it is a smooth, positive definite section of $S^2({}^{\text{sc}}T^*X)$, where ${}^{\text{sc}}T^*$ is the scattering cotangent bundle, dual to ${}^{\text{sc}}T$.

We remark that ${}^{\text{sc}}T$ and $\mathcal{V}_{\text{sc}}(X)$ are intrinsically associated to X as a smooth manifold with boundary: this is a question of seeing that the definitions are independent of choice of boundary defining function. Essentially

the same observation means that if X is any manifold with boundary, we can introduce ${}^{\text{sc}}T$ and $\mathcal{V}_{\text{sc}}(X)$; from the geometric point of view, a smooth metric on ${}^{\text{sc}}T^*X$ corresponds to (the big end of) an asymptotically conic metric on \mathring{X} .

The advantage of introducing ${}^{\text{sc}}T$ is that quantities naturally associated to the euclidean metric extend uniformly to the boundary. For example, the Hodge star operator gives an isometry

$$* : \Lambda^2 {}^{\text{sc}}T^* \longrightarrow {}^{\text{sc}}T^* \quad (2.16)$$

of bundles over X , including over the boundary.

2.2.1. b-structure . We pause to mention the parallel definitions of the algebra $\mathcal{V}_{\text{b}}(X)$ of vector fields that are tangent to ∂X , as this algebra will also play an important role later. We have $\mathcal{V}_{\text{sc}}(X) = x\mathcal{V}_{\text{b}}(X)$; there is a b-tangent bundle ${}^{\text{b}}T$ with the property that $C^\infty(X, {}^{\text{b}}T) = \mathcal{V}_{\text{b}}(X)$ and so on. In place of the local description (2.14) for ${}^{\text{sc}}T$ we have a local frame

$$x\partial_x, \partial_{y^1}, \partial_{y^2}$$

for ${}^{\text{b}}T$. We refer the reader to [Mel93] for a complete account.

2.3. Bogomolny equations. Let X be the radial compactification of \mathbb{R}^3 , equipped with the euclidean metric (regarded, as above, as a scattering metric on X), and fix a G principal-bundle $P \longrightarrow X$. Denote by ρ any boundary defining function for ∂X .

We consider here the definition of the monopole moduli spaces from the smooth point of view (i.e. building the boundary regularity of the data from the beginning, rather than assuming just the boundedness of the Yang–Mills–Higgs action) and recall some standard properties of these moduli spaces. There is nothing really new here: the definitions of framed and unframed moduli spaces are in [AH88] and the equivalence of the smooth definition with standard works on monopoles follows from the regularity results in [JT80].

Lemma 2.4. *The Bogomolny operator $*F_A - \nabla_A \Phi$ extends from the interior to define a smooth map*

$$\mathcal{B}(A, \Phi) : \mathfrak{C}(X, P) \longrightarrow \rho C^\infty(X, {}^{\text{sc}}T^*X \otimes \text{ad}(P)). \quad (2.17)$$

Proof. The proof is very easy, but instructive. First of all, any smooth 1-form extends canonically as an element of $\rho C^\infty(X, {}^{\text{sc}}T^*X)$. In particular,

$$(A, \Phi) \in \mathfrak{C} \Rightarrow \nabla_A \Phi \in \rho C^\infty(X, {}^{\text{sc}}T^*X \otimes \text{ad}(P)). \quad (2.18)$$

Next, suppose that p is a point of ∂X , and that (x, y_1, y_2) are adapted coordinates near p . If we suppose that $x^{-2}dx$, $x^{-1}dy_1$ and $x^{-1}dy_2$ are η -orthonormal at p , then

$$*(dx \wedge dy_1) = x^2 dy_2, \quad *(dy_1 \wedge dy_2) = dx \quad (2.19)$$

and it follows that $*$ maps $C^\infty(X, \Lambda^2 T^*X)$ into $\rho^2 C^\infty(X, {}^{\text{sc}}T^*X)$. Hence the result. \square

The fact that $*F(A)$ (as a section of ${}^{\text{sc}}T^*X$) is an order of magnitude smaller than $\nabla_A\Phi$ at ∂X has the following important consequences for the boundary behaviour of smooth monopoles.

Denote by \bar{A} and $\bar{\Phi}$ the restrictions of A and Φ to ∂X (cf. §2.1.1).

Proposition 2.5. *Suppose that $(A, \Phi) \in \mathfrak{C}$ satisfies*

$$\mathcal{B}(A, \Phi) = 0.$$

Then $\nabla_{\bar{A}}\bar{\Phi} = 0$.

Proof. We have seen that $*F_A = \mathcal{O}(\rho^2)$ whereas $\nabla_A\Phi = \mathcal{O}(\rho)$, as sections of ${}^{\text{sc}}T^* \otimes \text{ad}(P)$. By computations similar to those above, the coefficient of ρ in $\nabla_A\Phi$ is just $\nabla_{\bar{A}}\bar{\Phi}$, under the map

$$C^\infty(X, T^*X) \longrightarrow \rho C^\infty(X, {}^{\text{sc}}T^*X)$$

which regards an ordinary 1-form as a scattering 1-form which vanishes at the boundary. Thus the conclusion of the Proposition follows from the weaker assumption

$$\mathcal{B}(A, \Phi) \in \rho^2 C^\infty(X, {}^{\text{sc}}T^*X \otimes \text{ad}(P)). \quad \square$$

From now on we take $G = \text{SU}(2)$.

Definition 2.6. The *mass*, m , of the monopole (A, Φ) is the value of $|\bar{\Phi}|$. If $m > 0$, the *charge*, or *monopole number*, k , of (A, Φ) is the degree of the map $\bar{\Phi} : \partial X \longrightarrow \mathfrak{su}(2)$.

Proposition 2.5 implies that $|\bar{\Phi}|$ is constant. Hence the mass is well-defined. By trivializing P over ∂X , $\bar{\Phi}$ becomes a map into the sphere of radius m in $\mathfrak{su}(2)$. The degree of this map is independent of the trivialization, so the charge k is also well-defined.

From now on we assume $m > 0$, and usually $m = 1$. The latter is no loss of generality in euclidean space, for if $*_c$ denotes the Hodge star operator with respect to the rescaled euclidean metric $c^2\eta$, we have $*_c F(A) = c * F(A)$ and so if (A, Φ) is a monopole of mass m , then $(A, c\Phi)$ is a monopole of mass mc with respect to $c^2\eta$. If $m = 0$, then A is flat and $\Phi = 0$ [JT80, Theorem 10.3].

We now come to the definition of the unframed moduli space.

Definition 2.7. Fix an $\text{SU}(2)$ -bundle P over X , a positive mass m and a non-negative integer k . Then the moduli space of monopoles of mass m and charge k is defined to be the quotient

$$\mathcal{N}_{k,m} = \{(A, \Phi) \in \mathfrak{C}_{k,m}(X, P) : *F(A) = \nabla_A\Phi\} / \mathfrak{G}. \quad (2.20)$$

We note that the boundary regularity we have built in means that our configuration space \mathfrak{C} is smaller than the usual one where smoothness over \mathbb{R}^3 is assumed along with the finiteness of the action (1.2). Taubes [JT80, Theorem 10.5, p. 157] has however proved that any finite-action solution

of the Bogomolny equations extends, in suitable gauges, to the radial compactification X of \mathbb{R}^3 . Although the results explicitly given there give only uniform bounds on F_A and $\nabla_A \Phi$, bounds on their derivatives were obtained in Taubes's PhD thesis, and from such bounds it is possible to prove that there are gauges in which a monopole (A, Φ) extends smoothly to X . Thus our 'smooth' definition is equivalent to the usual one.

It is also well known that $\mathcal{N}_{k,m}$ is a smooth manifold of real dimension $4k - 1$. The reader is referred to [AH88] for more background on \mathcal{N}_k . For $k = 1$, every solution is, up to translations of \mathbb{R}^3 , gauge equivalent to the spherically symmetric solution found by 't Hooft and Polyakov [tH74, Pol74]. Thus $\mathcal{N}_{1,m} = \mathbb{R}^3$.

2.4. The boundary connection and symmetry-breaking. We saw in Proposition 2.5 that the boundary values $(\bar{A}, \bar{\Phi})$ of a smooth monopole satisfy $\nabla_{\bar{A}} \bar{\Phi} = 0$ on ∂X and in fact this conclusion follows from the weaker condition $\mathcal{B}(A, \Phi) = \mathcal{O}(\rho^2)$. In this section we take this discussion one stage further to determine \bar{A} up to gauge.

Before we go ahead, consider the action of Φ on the complexification $\mathfrak{p} := \text{ad}(P)_{\mathbb{C}}$ of $\text{ad}(P)$. At every point this action is diagonalizable and at points where $\Phi \neq 0$ there are precisely three distinct eigenvalues, 0 and $\pm i|\Phi|$. (We are using $G = \text{SU}(2)$ here.) Accordingly, at all points of X with $\Phi \neq 0$, and in particular in a neighbourhood of ∂X , we have a decomposition $\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_1$ where \mathfrak{p}_0 is the bundle annihilated by $\text{ad}(\Phi)$, while \mathfrak{p}_1 is its orthogonal complement.

Returning to monopoles, $\Phi \neq 0$ at infinity gives a reduction of the symmetry group from $\text{SU}(2)$ to the $\text{U}(1)$ subgroup stabilizing Φ .

Now if (A, Φ) is a solution of the Bogomolny equations, Taubes [JT80, §IV.10, Theorem 10.5] has proved not only the boundary regularity (in suitable gauges) but also *rapid off-diagonal decay*,

$$\text{Ad}(\Phi)F_A = \mathcal{O}(\rho^\infty), \text{ or equivalently } \text{Ad}(\Phi)\nabla_A \Phi = \mathcal{O}(\rho^\infty). \quad (2.21)$$

From this it follows that there exist local gauges in which Φ is diagonal and A is represented by a connection 1-form which is diagonal modulo terms of order ρ^∞ .

From our point of view, (2.21) can be proved by an iterative argument which combines the Bogomolny equations with the assumption that all data are smooth up to the boundary. We shall use (2.21) to simplify the formal construction in §3. We shall not give a complete proof of (2.21) here, but we shall prove the result to the next order. At the same time we shall determine the curvature of \bar{A} , which will also be important in the definition of the framed moduli space.

Let $p \in \partial X$ be any point, and let U be a product neighbourhood of p , with coordinates (x, y) , $0 \leq x < \delta$. We may choose 'boundary radial gauge' to write ∇_A in the form

$$\nabla_A = \bar{\nabla} + dx \otimes \partial_x + xb + \mathcal{O}(x^2) \quad (2.22)$$

where $b(y, dy)$ is an $\text{ad}(P)$ -valued 1-form on ∂X and we've written $\overline{\nabla}$ for $\nabla_{\overline{A}}$.

In this gauge, we may expand Φ in the form

$$\Phi = \overline{\Phi} + x\Phi_1(y) + \mathcal{O}(x^2). \quad (2.23)$$

Proposition 2.8. *Suppose that (A, Φ) have the above expressions in U and satisfy $*F(A) = \nabla_A \Phi$. Then we have*

$$\overline{\nabla} \overline{\Phi} = 0, \text{ and } \overline{\nabla} \Phi_1 = 0. \quad (2.24)$$

We also have

$$\text{ad}(\Phi_0)b = 0, \quad \text{ad}(\Phi_0)\Phi_1 = 0 \quad (2.25)$$

and

$$\dagger \overline{F} = \Phi_1, \quad (2.26)$$

where \dagger is the Hodge star operator with respect to the metric of the round unit sphere ∂X .

Proof. It is useful to keep in mind that with respect to the euclidean metric,

$$|dx| = \mathcal{O}(x^2), \quad |dy| = \mathcal{O}(x). \quad (2.27)$$

Then we compute

$$\begin{aligned} \nabla_A \Phi &= (\overline{\nabla} + dx \otimes \partial_x + xb)(\overline{\Phi} + x\Phi_1 + \mathcal{O}(x^2)) \\ &= \overline{\nabla} \overline{\Phi} \end{aligned} \quad (2.28)$$

$$+ x(\overline{\nabla} \Phi_1 + [b, \overline{\Phi}]) \quad (2.29)$$

$$+ dx(\Phi_1 + [a, \overline{\Phi}]) \quad (2.30)$$

$$+ \mathcal{O}(x^3). \quad (2.31)$$

We have set the terms out so that (2.28) is $\mathcal{O}(x)$, while (2.29) is $\mathcal{O}(x^2)$ and tangential, while (2.30) is $\mathcal{O}(x^2)$ and normal to ∂X . The expansion to $\mathcal{O}(x^3)$ of F_A is much simpler:

$$F_A = \overline{F} + \mathcal{O}(x^3) \quad (2.32)$$

so

$$*F_A = \dagger \overline{F} dx + \mathcal{O}(x^3). \quad (2.33)$$

We now simply equate coefficients. At $\mathcal{O}(x)$ we recover Proposition 2.5, which is the first of (2.24). The normal component at $\mathcal{O}(x^2)$ gives

$$\dagger \overline{F} = \Phi_1. \quad (2.34)$$

The tangential component at $\mathcal{O}(x^2)$ gives

$$\overline{\nabla} \Phi_1 + [b, \overline{\Phi}] = 0. \quad (2.35)$$

But the first term is a multiple of $\overline{\Phi}$, while the second is orthogonal to it. Hence both terms vanish, completing the proof. \square

2.5. Framed moduli space. We now define the framed moduli space \mathcal{M}_k . This will be a $U(1)$ bundle over \mathcal{N}_k and is in many ways the more natural object. Motivated by Propositions 2.5 and 2.8 we make the following definition:

Definition 2.9. A pair $(\bar{A}, \bar{\Phi})$ over ∂X is called *admissible* if

$$\bar{\nabla} \bar{\Phi} = 0, \bar{\Phi} \neq 0 \text{ and } \bar{\nabla} \dagger \bar{F} = 0, \quad (2.36)$$

where as above we have written $\bar{\nabla}$ for $\nabla_{\bar{A}}$ and \bar{F} for $F_{\bar{A}}$. We call the pair admissible *charge- k* boundary data if

$$\dagger \bar{F} = \frac{k}{2m} \bar{\Phi}. \quad (2.37)$$

where $m = |\bar{\Phi}|$.

Remark. Let $\bar{E} \rightarrow \partial X$ be the complex vector bundle associated by the fundamental representation of $SU(2)$. At each point p of ∂X , we have the eigenspaces $(L_{\pm})_p$ of $\bar{\Phi}_p$, viewed as an endomorphism of \bar{E}_p . By the first two conditions of (2.36), these eigenspaces patch together to form a pair of complex line bundles L_{\pm} over ∂X . Then $\bar{\Phi}$ acts as multiplication by $\pm im$ on L_{\pm} and \bar{A} preserves each of L_{\pm} . Thus there are $U(1)$ connections ∇_{\pm} on L_{\pm} such that

$$\bar{\nabla} = \text{diag}(\nabla_+, \nabla_-)$$

with respect to the isomorphism $E = L \oplus L^{-1}$, with $\bar{\Phi} = \text{diag}(im, -im)$. Then the curvature form of $\bar{\nabla}$ takes the form

$$\bar{F} = \text{diag}(f_+, f_-)(\dagger 1)$$

where f_{\pm} are imaginary functions on ∂X . In fact, L_+ and L_- are mutually dual, the connections ∇_{\pm} respect this and so $f_- = -f_+$. Since $\bar{\nabla}$ acts as d on the diagonal components of any endomorphism (these components being sections of a canonically trivial line bundle), the third part of (2.36) gives $df_{\pm} = 0$, so $f_{\pm} = \pm i\lambda$, for some real constant λ . By Chern–Weil theory, $\lambda = -\frac{1}{2}c_1(L_+)$, where we have identified the c_1 with an integer via the fundamental class of ∂X . This gives (2.37) if $c_1(L_{\pm}) = \mp k$.

The arrangement of signs comes from a standard computation which we recall here. We have

$$(F_A - *\nabla_A \Phi) \wedge *(F_A - *\nabla_A \Phi) = F_A \wedge *F_A + \nabla_A \Phi \wedge *\nabla_A \Phi - 2F_A \wedge \nabla_A \Phi.$$

Taking $(-1/2)\text{tr}$ of both sides and imposing the Bogomolny equations,

$$\|F_A\|^2 + \|\nabla_A \Phi\|^2 = - \int d \text{tr}(\Phi F_A) = - \int_{\partial X} \text{tr}(\bar{\Phi} \bar{F}).$$

If $\bar{\Phi} = \text{diag}(im, -im)$, $\dagger \bar{F} = (ik/2, -ik/2)$ it follows that

$$\|F_A\|^2 + \|\nabla_A \Phi\|^2 = 4\pi mk \quad (2.38)$$

and so $k > 0$ as required.

Given any choice of admissible boundary data, define the framed configuration space

$$\mathfrak{C}_0(X, P) = \{(A, \Phi) \in \mathfrak{C}(X, P) : (A, \Phi)|_{\partial X} = (\overline{A}, \overline{\Phi})\}, \quad (2.39)$$

the framed gauge group

$$\mathfrak{G}_0(X, P) = \{\gamma \in \mathfrak{G}(X, P) : \gamma|_{\partial X} = 1\}, \quad (2.40)$$

and the framed moduli space

$$\mathcal{M}_{k,m} = \{(A, \Phi) \in \mathfrak{C}_0 : \mathcal{B}(A, \Phi) = 0\} / \mathfrak{G}_0. \quad (2.41)$$

It is known that \mathcal{M}_k is non-compact smooth manifold of real dimension $4k$. The L^2 metric G_k from (1.7) gives \mathcal{M}_k the structure of a complete hyperkähler manifold. We shall say more about the local structure of \mathcal{M}_k in the next subsection.

We note also that there is a free isometric action of $U(1)$ on \mathcal{M}_k , with $\mathcal{M}_k / U(1) = \mathcal{N}_k$. This action is given explicitly by using Φ as a gauge transformation:

$$t \cdot (A, \Phi) = e^{t\Phi}(A, \Phi), \quad t \in \mathbb{R}; \quad (2.42)$$

the point is that the boundary data $(\overline{A}, \overline{\Phi})$ are fixed by this action. If $0 < t < 2\pi/m$, then $e^{t\Phi}$ is not in \mathfrak{G}_0 so (2.42) is a non-trivial action. The action is periodic, however, of period $2\pi/m$ and so (2.42) defines an action of the circle $\mathbb{R}/(2\pi\mathbb{Z}/m)$ on $\mathcal{M}_{k,m}$. It is clear that the quotient is \mathcal{N}_k .

Remark. Although our definition of \mathcal{M}_k depends on a choice of admissible boundary data, any two choices lead to the same moduli space. To be more precise, let \mathfrak{C}'_0 and \mathcal{M}'_k be respectively the configuration space and the moduli space defined by replacing $(\overline{A}, \overline{\Phi})$ by a different choice $(\overline{A}', \overline{\Phi}')$ of admissible boundary data. Then we claim that there is a principal $U(1)$ -set of natural identifications $\gamma_t : \mathcal{M}_k \simeq \mathcal{M}'_k$, $t \in \mathbb{R}/(2\pi\mathbb{Z}/m)$, the $U(1)$ action being given by composition with the above $U(1)$ -action on either of \mathcal{M}_k or \mathcal{M}'_k .

The main point is that since \overline{A}' and \overline{A} have the same curvature and the base is simply connected, \overline{A} and \overline{A}' are gauge-equivalent by some gauge transformation \overline{g} . Clearly \overline{g} is unique up to composition with gauge transformations which preserve \overline{A} , and this group is isomorphic to the $U(1)$ generated by $\exp(t\Phi)$. If $g \in \mathfrak{G}$ is any extension of \overline{g} , then

$$(A, \Phi) \longmapsto g(A, \Phi)$$

defines a diffeomorphism from \mathfrak{C}_0 to \mathfrak{C}'_0 and this induces the desired identification of \mathcal{M}_k with \mathcal{M}'_k .

2.6. The local structure of \mathcal{M}_k . Let (A, Φ) be a solution of the Bogomolny equations representing a point m of \mathcal{M}_k . The infinitesimal structure of \mathcal{M}_k is described by the *deformation complex*

$$0 \longrightarrow \text{Lie}(\mathfrak{G}_0) \xrightarrow{d} T_{(A, \Phi)} \mathfrak{C}_0 \xrightarrow{DB} C^\infty(X, {}^{sc}T^* \otimes \text{ad}(P)) \longrightarrow 0. \quad (2.43)$$

Here

$$\mathrm{Lie}(\mathfrak{G}_0) = \rho C^\infty(X, \mathrm{ad}(P)), \quad (2.44)$$

$$T_{(A,\Phi)}\mathfrak{C}_0 = \rho C^\infty(X, \mathrm{ad}(P)) \oplus \rho^2 C^\infty(X, {}^{\mathrm{sc}}T^* \otimes \mathrm{ad}(P)) \quad (2.45)$$

and

$$d_{(A,\Phi)}(\xi) = (-d_A u, -\mathrm{ad}(\Phi)u). \quad (2.46)$$

The cohomology in the middle of (2.43) is the (virtual) tangent space of \mathcal{M}_k at m , and may be identified with the true tangent space if $D\mathcal{B}$ is surjective and the action of \mathfrak{G}_0 is free. Fortunately these conditions are always satisfied for monopoles.

As in the introduction, the linearization $D\mathcal{B}$ of the Bogomolny equation combines with the Coulomb gauge fixing operator $d_{(A,\Phi)}^*$ to give the operator $L_{(A,\Phi)}$ of (1.5). One shows [Kot15c] that the cohomology in the middle of (2.43) is isomorphic to the null-space of $L_{(A,\Phi)}$,

$$L_{(A,\Phi)} : \rho^2 C^\infty(X, \Lambda \otimes \mathrm{ad}(P)) \longrightarrow \rho^2 C^\infty(X, \Lambda \otimes \mathrm{ad}(P)), \quad (2.47)$$

where $\Lambda = {}^{\mathrm{sc}}\Lambda^1 \oplus {}^{\mathrm{sc}}\Lambda^0$.

Remark. The reason why the domain is different from (2.45) is that the coefficient of ρ in ϕ is fixed by Proposition 2.5 (by the Bogomolny equations). In particular any solution of the linearized Bogomolny equations in the domain (2.45) is actually $\mathcal{O}(\rho^2)$.

By a slight abuse of notation, set

$$T_{(A,\Phi)}\mathcal{M}_k = N(L_{A,\Phi}), \quad (2.48)$$

the null-space of $L_{A,\Phi}$ with domain (2.47).

One obtains the dimension of \mathcal{M}_k and its smoothness by combining an index theorem for (2.47) with the Weitzenböck formula

$$L_{A,\Phi} L_{A,\Phi}^* = \nabla_A^* \nabla_A + \mathrm{ad}(\Phi^*) \mathrm{ad}(\Phi). \quad (2.49)$$

This formula is valid whenever the base metric is Ricci-flat and $\mathcal{B}(A, \Phi) = 0$. The index has been studied by Taubes [Tau83] and Kottke [Kot15c] and the result is that there is a Fredholm framework for $L_{A,\Phi}$ making it surjective (between spaces which are suitable completions of those in (2.47)) with index $4k$.

The same analytic framework allows us to prove that \mathcal{M}_k is a smooth manifold. One proves first that for all sufficiently small $(\tilde{a}, \tilde{\phi}) \in \mathfrak{C}_0$, there is an (a, ϕ) gauge related to $(\tilde{a}, \tilde{\phi})$ in Coulomb gauge with respect to (A, Φ) ,

$$d_{(A,\Phi)}^*(a, \phi) = 0. \quad (2.50)$$

Hence any nearby point $m' \in \mathcal{M}_k$ is represented by $(A + a, \Phi + \phi)$ where

$$L_{(A,\Phi)}(a, \phi) = -Q(a, \phi), \quad (2.51)$$

and

$$Q(a, \phi) = *[a, a] - [a, \phi] \quad (2.52)$$

is the nonlinear part of \mathcal{B} . It follows from the surjectivity of $L_{(A,\Phi)}$ and the implicit function theorem that the set of small solutions to (2.51) is the graph of a nonlinear operator from $T_m\mathcal{M}_k$ to its L^2 -orthogonal complement in $\rho^2 C^\infty(X, \Lambda \otimes \text{ad}(P))$.

2.7. The centre of a monopole. The unframed moduli space for monopoles of charge 1 is diffeomorphic to \mathbb{R}^3 ; the interpretation of this is that given the 't Hooft monopole m_0 centred at the origin of \mathbb{R}^3 , every other monopole of charge 1 is a translation of m_0 by some vector in \mathbb{R}^3 .

For $k > 0$, one cannot generally distinguish k individual charge-1 monopoles with definite centres, but every $m \in \mathcal{M}_k$ does have a well-defined centre. To explain the definition, revert to euclidean coordinates $z \in \mathbb{R}^3$, and observe that the Higgs field has an expansion of the form

$$\Phi = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \left(m - \frac{k}{2|z|} - \frac{k}{2} \frac{v \cdot z}{|z|^3} + \mathcal{O}(|z|^{-3}) \right) \quad (2.53)$$

relative to the decomposition of $\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_1$. Here $v \in \mathbb{R}^3$ is some vector.

Definition 2.10. For $m \in \mathcal{M}_k$ with Φ given by (2.53), the *centre* of m is defined to be the vector v/m (relative to the origin of the z coordinates). The moduli space of monopoles with centre at 0 is denoted \mathcal{M}_k^c .

The definition can be motivated by considering the change in v when we translate the monopole. More precisely, if $c \in \mathbb{R}^3$, consider (A_c, Φ_c) , the result of pulling back by the translation $z \mapsto z + c$. Then the expansion of Φ_c is

$$\Phi_c(z) = \Phi(z - c) = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \left(m - \frac{k}{2|z|} - \frac{k}{2} \frac{(v + c) \cdot z}{|z|^3} + \mathcal{O}(|z|^{-3}) \right) \quad (2.54)$$

Thus our definition is consistent with the translation action on \mathbb{R}^3 , for the centre of m should be translated by c if the whole monopole is translated by c .

Remark. We note without proof that the $U(1)$ action on \mathcal{M}_k is triholomorphic. Viewing the associated hyperkähler moment map μ as a map to \mathbb{R}^3 , $\mu(m)$ is equal to the centre of m .

Remark. From the previous remark $\mathcal{M}_k^0 := \mathcal{M}_k^c / U(1)$ is a hyperkähler quotient of \mathcal{M}_k , hence a hyperkähler manifold of dimension $4k - 4$. As discussed in [AH88, p. 20], its universal cover $\widetilde{\mathcal{M}}_k^0$ is a k -fold cover, called the moduli space of strongly centred monopoles. This appears in the decomposition

$$\widetilde{\mathcal{M}}_k = \mathbb{R}^3 \times S^1 \times \widetilde{\mathcal{M}}_k^0 \quad (2.55)$$

as a riemannian product [AH88, p. 21]. We shall discuss this at the infinitesimal level in the next section.

2.8. Structure of the tangent space. Although we shall not make great use of it, recall that $T_m\mathcal{M}_k$, defined as in (2.48), has a quaternionic structure. For this, pick any oriented orthonormal basis (e_1, e_2, e_3) of \mathbb{R}^3 , identify it with the corresponding frame for $T^*\mathbb{R}^3$, and write any element of $T_m\mathcal{M}_k$ in the form

$$a_0 = \phi, \quad a = \sum a_j e_j$$

where $a_0, \dots, a_3 \in \rho^2 C^\infty(\mathbb{R}^3; \text{ad}(P))$. This decomposition identifies the domain of $L_{(A, \Phi)}$ in (2.47) with $\rho^2 C^\infty(\mathbb{R}^3; \text{ad}(P)) \otimes_{\mathbb{R}} \mathbb{R}^4$ and identifying \mathbb{R}^4 with the quaternions we have endowed this domain with the structure of a quaternionic vector space. One checks that $L_{(A, \Phi)}$ commutes with this quaternionic structure, and it follows that $T_m\mathcal{M}_k = N(L_{(A, \Phi)})$ inherits a quaternionic structure.

Our next task is to describe the subspace of $T_m\mathcal{M}_k$ corresponding to the action of translations or \mathbb{R}^3 and changes of framing. This will turn out to be a quaternionic subspace of $T_m\mathcal{M}_k$, and its orthogonal complement is identifiable with the elements of $T_m\mathcal{M}_k$ which are $\mathcal{O}(\rho^3)$ at ∂X . The decomposition is the infinitesimal version of the riemannian splitting of the universal cover of \mathcal{M}_k as a product of $\mathbb{R}^3 \times S^1$ with the moduli space of strongly centred monopoles [AH88].

For any infinitesimal gauge transformation $\xi \in \text{Lie}(\mathfrak{G})$, the corresponding tangent vector at (A, Φ) in \mathfrak{C} is $d_{A, \Phi}\xi$. Taking $\xi = \Phi$, we get the tangent vector

$$\tau_0 = (-\nabla_A \Phi, 0) \tag{2.56}$$

which is easily verified to lie in $T_m\mathcal{M}_k$. (One has to check only that $d_{A, \Phi}^* \tau_0 = 0$ and that $\tau_0 = \mathcal{O}(\rho^2)$.) For an orthonormal basis e_i of \mathbb{R}^3 , set

$$\tau_i = (\iota_{e_i} F_A, \nabla_{e_i} \Phi); \tag{2.57}$$

this is the element of $T_{(A, \Phi)}\mathcal{M}_k$ corresponding to the infinitesimal translation $z \mapsto z + te_i$ of (A, Φ) . Note that this is nothing other than the derivative of (A, Φ) with respect to e_i , using the connection A itself to lift e_i to P .

We note that the τ_a , $a = 0, 1, 2, 3$ span a quaternionic subspace of $T_{A, \Phi}\mathcal{M}_k$: τ_i is obtained from τ_0 by applying the i -th complex structure. We leave the verification that $L_{A, \Phi}\tau_a = 0$ to the reader.

Proposition 2.11. *Let (A, Φ) represent an element of \mathcal{M}_k . There is an orthogonal direct-sum decomposition*

$$T_m\mathcal{M}_k = \langle \tau_0, \dots, \tau_3 \rangle \oplus T'_m\mathcal{M}_k \tag{2.58}$$

where

$$T'_m\mathcal{M}_k = \{u \in C^\infty(X, \Lambda \otimes \text{ad}(P)) : |u| = \mathcal{O}(\rho^3)\} \tag{2.59}$$

Moreover, if

$$v = \sum \lambda_a \tau_a, \tag{2.60}$$

we have

$$\|v\|_{L^2}^2 = 2\pi m k (\lambda_0^2 + \dots + \lambda_3^2). \tag{2.61}$$

Remark. If $m \in \mathcal{M}_k^c$, then T'_m is identifiable with $T_m \mathcal{M}_k^0$, where we have made no distinction between m and the point it represents in $\mathcal{M}_k^c/U(1)$. More generally, T'_m is the subspace of infinitesimal changes in m which keep its centre and framing fixed.

Proof. We verify (2.61) first. For τ_0 , we have by (2.38)

$$\|\tau_0\|^2 = \int |\nabla_A \Phi|^2 = 2\pi m k. \quad (2.62)$$

One verifies $\|\tau_i\|^2 = 2\pi m k$ either by hand, or by noting that $\tau_i = I_i \tau_0$, where I_1, I_2, I_3 are the three complex structures on $T_m \mathcal{M}_k$, which are isometries of this space.

Next, consider any solution,

$$L_{(A,\Phi)} u = 0, \quad |u| = \mathcal{O}(\rho^2). \quad (2.63)$$

By boundary regularity for $L_{A,\Phi}$, the off-diagonal components of u are $\mathcal{O}(\rho^\infty)$, and so if u_0 is the diagonal \mathfrak{p}_0 component of u , we have

$$L_{A,\Phi} u_0 = \mathcal{O}(\rho^\infty). \quad (2.64)$$

The action of $L_{A,\Phi}$ on $\Lambda \otimes \mathfrak{p}_0$ is, up to rapidly decreasing terms at ∂X , that of the model operator

$$L = \begin{bmatrix} *d & -d \\ -d^* & 0 \end{bmatrix}, \quad (2.65)$$

(cf. Appendix C) and so the asymptotic expansion of u_0 in (2.64) is a sum of homogeneous solutions f of $Lf = 0$. Since $L^2 = \Delta$, the Laplacian on Λ , if f is $\mathcal{O}(\rho^2)$, then the 4 components of f all have to be of the form $\langle c, z \rangle / |z|^3$, for some constant $c \in \mathbb{R}^3$. Then for $Lf = 0$, we need

$$f_0 = \begin{bmatrix} 0 \\ z/|z|^3 \end{bmatrix}, \quad \text{or } f_c = \begin{bmatrix} \langle c, z \rangle / |z|^3 \\ -(c \wedge z) / |z|^3 \end{bmatrix}. \quad (2.66)$$

On the other hand, each of these solutions can be continued over X as a linear combination of the τ_a . It follows that we have an exact sequence

$$0 \longrightarrow T_{(A,\Phi)} \mathcal{M}_k^c \longrightarrow T_{(A,\Phi)} \mathcal{M}_k \longrightarrow \mathbb{R}^4 \longrightarrow 0 \quad (2.67)$$

where $T_{(A,\Phi)} \mathcal{M}_k^c$ is as in (2.59).

For the orthogonality, note that we can rewrite

$$\sum \lambda_a \tau_a = L_{(A,\Phi)} \begin{bmatrix} \lambda_0 \Phi \\ \underline{\lambda} \Phi \end{bmatrix} = L_{(A,\Phi)}^* \begin{bmatrix} \lambda_0 \Phi \\ \underline{\lambda} \Phi \end{bmatrix} \quad (2.68)$$

where $\underline{\lambda}$ is the euclidean vector $(\lambda_1, \lambda_2, \lambda_3)$. (Similarly, the model solution f can be written

$$f = L \begin{bmatrix} \lambda_0 |z|^{-1} \\ \underline{\lambda} |z|^{-1} \end{bmatrix}.) \quad (2.69)$$

Now if $v \in T_{(A,\Phi)} \mathcal{M}_k^c$, we compute

$$(v, \sum \lambda_a \tau_a) = \int_X \langle v, L_{(A,\Phi)}^* (\lambda \Phi) \rangle \quad (2.70)$$

Restricting to the sphere $\rho = \delta$ we are left with a boundary term of the form

$$\int_{\rho=\delta} \langle v, \lambda \Phi \rangle \quad (2.71)$$

because $L_{(A,\Phi)}v = 0$. Because $|v| = \mathcal{O}(\rho^3)$, $|\lambda \Phi| = \mathcal{O}(1)$ but the area of $\rho = \delta$ is $4\pi\delta^{-2}$, this boundary term goes to zero as $\delta \rightarrow 0$. This completes the proof of the proposition. \square

2.9. Dirac monopoles. If we take $G = \mathrm{U}(1)$, the Bogomolny equations reduce to equations studied by Dirac in relation to ordinary magnetic monopoles. There are no non-trivial solutions on \mathbb{R}^3 without singularities, but these Dirac monopoles are also a key ingredient in the gluing theorem that will be discussed below. We therefore give a quick account geared to our later applications. In particular we give a careful discussion of moduli spaces of Dirac monopoles with fixed singularities. Although the discussion can easily be extended to other 3-manifolds [Oli14] we shall confine ourselves to the case of \mathbb{R}^3 .

Let U be an open set of \mathbb{R}^3 . Let $Q \rightarrow U$ be a $\mathrm{U}(1)$ principal bundle with connection a and let ϕ be section of the adjoint bundle. Since this is canonically trivial, ϕ can be identified canonically with an imaginary function on U . Similarly the curvature F_a is canonically an imaginary closed 2-form on U and $(i/2\pi)F_a$ represents $c_1(Q)^2$.

Suppose that (a, ϕ) satisfies the Bogomolny equations

$$*F_a = \nabla_a \phi = d\phi. \quad (2.72)$$

Since $dF_a = 0$ we deduce from this that ϕ is harmonic,

$$\Delta \phi = 0 \text{ in } U. \quad (2.73)$$

Conversely, given an imaginary harmonic function on U with *integral periods* in the sense that

$$\frac{i}{2\pi} \int_{\Sigma} *d\phi \in \mathbb{Z} \text{ for all } \Sigma \in H_2(U, \mathbb{Z}) \quad (2.74)$$

there exists a pair (Q, a) if a $\mathrm{U}(1)$ bundle and connection with

$$F_a = *d\phi. \quad (2.75)$$

In this case the gauge group is $C^\infty(U, \mathrm{U}(1))$ and if γ is an element of the gauge group,

$$\gamma \cdot (a, \phi) = (a - (d\gamma)\gamma^{-1}, \phi) \quad (2.76)$$

We shall reserve the term ‘Dirac monopole’ for the case that $U = \mathbb{R}^3 \setminus \{z_1, \dots, z_N\}$ and ϕ has the simplest possible singularities at the points z_j .

²In this section we are using (a, ϕ) for $\mathrm{U}(1)$ -valued monopole data, rather than infinitesimal deformations of $\mathrm{SU}(2)$ -monopoles. We hope that no confusion will result from this

In keeping with the general philosophy of this paper, we shall replace this punctured non-compact manifold by a compact manifold with boundary

$$D := [\overline{\mathbb{R}^3}; z_1, \dots, z_N]. \quad (2.77)$$

This is the real blow-up at the points z_j (assumed distinct) of the radial compactification X of \mathbb{R}^3 . Thus D has $N + 1$ boundary hypersurfaces, each of which is diffeomorphic to the 2-sphere \mathbb{S}^2 , and which will be denoted by S_1, \dots, S_N and S_∞ . Here S_j is the boundary hypersurface introduced by blowing up z_j while S_∞ is the original boundary of X . For each j , S_j is canonically identified with the sphere bundle of $T_{z_j}X$, $S_j = (T_{z_j}X \setminus \{0\})/\mathbb{R}^+$, where \mathbb{R}^+ is the multiplicative group of positive real numbers. Thus D is the natural domain in which polar coordinates have been introduced at each of the z_j . Indeed, the euclidean distance r_j from z_j lifts to D to be a (smooth) boundary defining function for S_j , which we denote without change of notation. We continue to denote the boundary defining function of S_∞ by ρ ; as before, this can be taken to be the reciprocal of the euclidean distance from any given point of \mathbb{R}^3 .

The Euclidean metric has the form

$$\frac{d\rho^2}{\rho^4} + \frac{h_{\mathbb{S}^2}}{\rho^2}, \quad \text{resp.} \quad dr_j^2 + r_j^2 h_{\mathbb{S}^2}, \quad (2.78)$$

in a product neighborhood of S_∞ or S_j , respectively. Thus it is natural to introduce a rescaled *conic* tangent bundle cTD ,

$${}^cTD = (r_1 \dots r_N)^{-1} \rho^b TD. \quad (2.79)$$

The space of all smooth sections $C^\infty(D, {}^cTD)$ is the space $\mathcal{V}_c(D)$ of tangent vector fields of bounded length with respect to the lifted euclidean metric; equivalently this lifted metric is a smooth and everywhere positive-definite section of $S^2 {}^cT^*D$. An important point about $\mathcal{V}_c(D)$ is that it is not an algebra (in contrast to \mathcal{V}_b and \mathcal{V}_{sc}). It is perhaps best thought about in terms of rescaled b-vector fields via (2.79).

Definition 2.12. Let D be as above and let $Q \rightarrow D$ be any $U(1)$ principal bundle. A *Dirac monopole* on D is a pair (a, ϕ) where a is a smooth connection on Q , $\phi \in C^\infty(\mathring{D})$ and the Bogomolny equations (2.72) are satisfied in \mathring{D} .

We shall now classify Dirac monopoles in the sense of this definition.

Proposition 2.13. Let $Q \rightarrow D$ and (a, ϕ) be a Dirac monopole configuration. Then there exists a constant m and integers k_j such that

$$\phi = i \left(m - \sum_{j=1}^N \frac{k_j}{|z - z_j|} \right) \quad (2.80)$$

Proof. Pick any one of the z_j and for simplicity write $r_j = r$, $y = (y^1, y^2)$ for local coordinates on S_j . Then F_a is a smooth linear combination of $dr \wedge dy$ and $dy^1 \wedge dy^2$ and so $*F_a$ is a smooth linear combination of dy and $r^{-2}dr$.

On the other hand, ϕ is harmonic away from $r = 0$ and so (cf. Appendix C) it is a linear combination of terms of the form $p(z - z_j)|z - z_j|^{2\nu+1}$, where p is a harmonic polynomial, homogeneous of degree ν . Given that $\partial_r \phi = \mathcal{O}(r^{-2})$, it follows that we can only have $\nu = 0$, so

$$\phi(z) \sim \frac{\lambda_j}{|z - z_j|} + \mathcal{O}(1) \quad (2.81)$$

for some constant λ_j . For ϕ to have an integral period around S_j , we need $\lambda_j = -ik_j/2$, k_j an integer. Repeating the argument at all of the z_j , we find

$$\phi(z) + i \sum \frac{k_j}{2|z - z_j|} = h(z) \quad (2.82)$$

where h is a bounded harmonic function on \mathbb{R}^3 . Thus h must be a constant and ϕ has the given form. \square

The $U(1)$ -bundle $Q \rightarrow D$ is classified up to isomorphism by its Chern class $c_1(Q) \in H^2(D; \mathbb{Z}) = \mathbb{Z}^N$, which is determined by the values $k_i := c_1(Q)[S_i]$, $i = 1, \dots, N$. This is the significance of the integers k_i in (2.80). Note that we have the relation

$$k_\infty := c_1(Q)[S_\infty] = k_1 + \dots + k_N.$$

We refer to (k_1, \dots, k_N) as the *charge* of Q , and hence of the monopole. If a is a smooth connection on Q satisfying the Bogomolny equations for some ϕ , we call it a *Dirac connection* on Q . Proposition 2.13 shows that if a is a Dirac connection on Q , then a determines a Higgs field ϕ uniquely up to the addition of a constant, such that (a, ϕ) satisfies the Bogomolny equations (2.72).

We now consider the framed moduli space of Dirac connections on a given $U(1)$ -bundle Q .

Definition 2.14. A *Dirac framing* of Q is a choice of boundary connection \bar{a} on $Q|_{\partial D}$, with locally constant curvature. Similarly if $\partial' D$ is a union of boundary components of D , a *partial Dirac framing* of Q over $\partial' D$ is a choice of boundary connection \bar{a} with locally constant curvature on $Q|_{\partial' D}$.

Locally constant curvature here means that on each boundary component S , the curvature $F_{\bar{a}}$ is a constant multiple of the area form on S .

Denote by $\mathfrak{G}_0(Q)$ the group of $U(1)$ -gauge transformations of D which are the identity over ∂D . It is convenient to introduce the notation $j : \partial D \rightarrow D$ for the boundary inclusion.

Proposition 2.15. *Let \bar{a} be a Dirac framing of Q . The moduli space*

$$\{\text{Dirac connections on } Q, \text{ framed by } \bar{a}\} / \mathfrak{G}_0(Q) \quad (2.83)$$

is diffeomorphic to the principal $U(1)$ -set

$$C_1 \times \cdots \times C_N \quad (2.84)$$

where each C_j is the $U(1)$ group of gauge transformations which fix $\bar{a}|_{S_j}$.

Proof. Suppose for the moment that a is a Dirac connection with $j^*(a) = \bar{a}$. Let b be any other such connection. By definition they have the same curvature and the same restriction to ∂D , so $d(b-a) = 0$, $j^*(a-b) = 0$. Since $H^1(D; i\mathbb{R}) = 0$, there exists an imaginary function u such that $b = a + du$, and $dj^*(u) = 0$. u is determined up to the addition of a constant, and we use this to fix $u|_{S_\infty} = 0$. Thus b determines a collection of phases $e^{u_j} = e^u|_{S_j}$; since u is locally constant on ∂D , e^{u_j} is constant on S_j . It is clear that this map is a diffeomorphism between (2.83) and (2.84), given the choice of a .

Suppose now that a' is a different basepoint in the space of Dirac connections framed by \bar{a} . Then by the argument just used, there is a function v such that $a' = a + dv$, $v|_{S_\infty} = 0$. Then $b - a' = d(u - v)$ so the phases e^{u_j} are replaced by $e^{u_j - v_j}$ where $v_j = v|_{S_j}$.

It remains only to verify that there does exist a Dirac connection a with $j^*(a) = \bar{a}$. But this is straightforward: smooth Dirac connections on Q do exist, by reversing the argument of Proposition 2.13, using the smoothness of $*d\phi$. Let a_0 be any such Dirac connection and define $\bar{a}_0 = j^*(a_0)$. Then \bar{a}_0 is a Dirac framing of ∂D . But any two Dirac framings are gauge-equivalent because they have the same curvature and ∂D is simply connected (cf. §2.5.) Extend this gauge transformation smoothly from the boundary, and call this κ . Then $\kappa^*(a_0)$ is a Dirac connection on D and its restriction to the boundary is now \bar{a} . \square

If instead we have a partial Dirac framing over $\partial' D$, a union of m connected components of the boundary, then the same argument gives the moduli space of partially framed Dirac connections on Q as being $(S^1)^m/U(1)$.

Proposition 2.16. *Let $I \subset \{1, \dots, N\}$. Then the moduli space of Dirac connections framed at $\bigcup_{i \in I} S_i \cup S_\infty$ is a principal $U(1)^{|I|}$ -space, consisting of the product $\prod_{i \in I} C_i$.*

In particular,

Corollary 2.17. *Let $Q \rightarrow D$ be a $U(1)$ bundle, with a given Dirac framing over S_∞ and set*

$$\mathfrak{G}_\infty(Q) = \{\gamma \in \mathfrak{G}(Q) : \gamma|_{S_\infty} = 1\}. \quad (2.85)$$

Then the moduli space

$$\{\text{Dirac connections } a \text{ on } Q, \text{ framed over } S_\infty\} / \mathfrak{G}_\infty(Q) \quad (2.86)$$

is a point.

Remark. In view of this Corollary, we see that the moduli space of Dirac connections on Q without any framing should be regarded as the space

$\{*\}/U(1)$, with formal dimension -1 . For this reason, our spaces of Dirac monopoles will always be framed at least over S_∞ .

In §4 we will let the points (z_1, \dots, z_N) vary and consider the larger moduli space with these points as well as the framings as moduli.

2.9.1. Dirac $SU(2)$ connections. In the next section we shall need to consider $SU(2)$ connections built from Dirac connections in the following trivial way. Let $Q \rightarrow D$ be a $U(1)$ bundle with a Dirac connection a . Let $\iota : U(1) \rightarrow SU(2)$ be a given embedding. Then the $SU(2)$ bundle

$$P = Q \times_{U(1)} SU(2) \quad (2.87)$$

is called a Dirac $SU(2)$ bundle and the connection on P induced by ι from a is called a Dirac $SU(2)$ connection.

3. FORMAL 1-PARAMETER FAMILIES

In this section we make a start on proving our first main result, Theorem 1.2. So fix a configuration

$$\underline{\zeta} = (\zeta_1, \dots, \zeta_N), \quad \zeta_i \neq \zeta_j \neq 0, \quad \sum_i |\zeta_i|^2 = 1$$

of nonzero points in \mathbb{R}^3 up to scaling, and a collection of monopoles (A_j, Φ_j) , where (A_0, Φ_0) represents a point of \mathcal{M}_{k_0} and for $j = 1, \dots, N$, (A_k, Φ_k) represents a point of $\mathcal{M}_{k_j}^c$. Here $k_0 \geq 0$ and $k_j \geq 1$ for $j = 1, \dots, N$. Given such data, we shall construct a 1-parameter family of $(A(\varepsilon), \Phi(\varepsilon)) \in \mathfrak{C}_k$ for $0 < \varepsilon < \varepsilon_0$ which are very good approximate solutions to the Bogomolny equations,

$$\mathcal{B}(A(\varepsilon), \Phi(\varepsilon)) = \mathcal{O}(\varepsilon^\infty) \quad (3.1)$$

We shall show how to solve away the error in (3.1) in §5.

The construction of $(A(\varepsilon), \Phi(\varepsilon))$ given here takes place on a manifold with corners, Z , referred to as the ‘gluing space’, equipped with a map $\varrho : Z \rightarrow [0, \infty)$. For positive ε , the fibre $\varrho^{-1}(\varepsilon)$, is canonically a copy of $X = \overline{\mathbb{R}^3}$, but $\varrho^{-1}(0)$ is a more complicated manifold with ‘normal crossings’. In our construction $(A(\varepsilon), \Phi(\varepsilon))$ will be the restriction to $\varrho^{-1}(\varepsilon)$ of a monopole configuration (A, Φ) on Z which is smooth in the interior and has only conormal singularities at the union of boundary hypersurfaces $\varrho^{-1}(0)$.

3.1. Gluing space. The gluing space Z , which will support $(A(\varepsilon), \Phi(\varepsilon))$, is constructed in two steps. Let $X = \overline{\mathbb{R}^3}$ and $Z_0 = X \times [0, \infty)_\varepsilon$. Define

$$Z_1 := [Z_0; \partial X \times \{0\}],$$

the real blow-up (cf. [Mel93]) of Z_0 in the corner $X \times \{0\}$. The new boundary hypersurface, denoted by D_1 , is by definition the projectivization of the inward pointing normal bundle of $\partial X \times \{0\}$ in Z_0 , hence diffeomorphic to $\mathbb{S}^2 \times [0, \infty]_s$, where $s = \varepsilon/x$ is the ratio of ε and a fixed boundary defining function x for X . Over the interior of D_1 , there is a natural Euclidean

coordinate $\zeta = \varepsilon z : \mathring{D}_1 \cong \mathbb{R}^3 \setminus \{0\}$, where z is the Euclidean coordinate on \mathring{X} .

The paths $z = \zeta_j/\varepsilon$ in $\mathring{Z}_1 \cong \mathbb{R}^3 \times (0, \infty)$ have well-defined limits at D_1 ; these are simply the points $\zeta = \zeta_j \in \mathring{D}_1$. The *gluing space* is defined by blowing up these points:

$$Z := [Z_1; \{\zeta_1, \dots, \zeta_N\}].$$

We denote the hypersurface which arises when ζ_j is blown up by X_j , $j = 1, \dots, N$, and denote lift of D_1 by D . Observe that $D_1 \cong \partial X \times [0, \infty]$ may be identified with $[\overline{\mathbb{R}^3}; 0]$, and then $D \cong [\overline{\mathbb{R}^3}; \zeta_0, \dots, \zeta_N]$, where $\zeta_0 = 0$. The lift of the original faces $X \times \{0\}$ and $\partial X \times [0, \infty)$ of Z_0 are denoted by X_0 and B , respectively. We set $S_j = X_j \cap D$ for $j = 0, \dots, N$ and $S_\infty = B \cap D$.

Define maps

$$\varrho : Z \longrightarrow [0, \infty), \quad \pi_X : Z \longrightarrow X,$$

by the composite of the blow-down maps to Z_0 with the projection to $[0, \infty)$ and X , respectively. These are easily seen to be b-fibrations [Mel92].

Boundary defining functions will be denoted by ρ_j for X_j , $j = 0, \dots, N$, and by ρ_D and ρ_B for the hypersurfaces D and B . We will sometimes write $\rho_X = \rho_0 \cdots \rho_N$ for the product of the defining functions for all the X_j . We confuse ε with its pull back to Z and assume the ρ 's are defined so that

$$\varepsilon = \rho_D \rho_X = \rho_D \rho_0 \cdots \rho_N.$$

As the projectivization of the inward pointing normal bundles to the ζ_j , the faces X_j have the structure of radially compactified affine spaces of real dimension 3. This can be understood geometrically as follows. If the path $z = \zeta_j/\varepsilon$ in \mathring{Z}_0 defined above is modified by

$$z = \zeta_j/\varepsilon + v \tag{3.2}$$

for a fixed vector in \mathbb{R}^3 , then these two paths have the same limits $\zeta = \zeta_j$ in Z_1 but their lifts have distinct limits on the face X_j in Z . Conversely, any two distinct points on X_j are the endpoints of some pair of paths which differ from one another by a fixed vector in \mathbb{R}^3 for $\varepsilon > 0$. Since a non-zero normal vector at ζ_j determines a point in X_j , the tangent vector to the lifted curve $z(\varepsilon) = \zeta_j/\varepsilon$ singles out a point of X_j and this gives its interior the structure of a vector space rather than just an affine space. Thus (3.2) gives an identification

$$X_j \cong \overline{\mathbb{R}^3} \tag{3.3}$$

for each j .

3.2. Metric structure. The metric structure we consider on Z is induced by the pullback $\pi_X^* g$ of the Euclidean metric on X . In order to interpret this correctly, we need to define the appropriate rescaled tangent bundle on Z .

We start with the bundle bTZ , defined so that the space of smooth sections $C^\infty(Z; {}^bTZ)$ is equal to the space of vector fields $\mathcal{V}_b(Z)$ which are

tangent to all boundary hypersurfaces. The subspace of *vertical vector fields*

$$\mathcal{V}_\varrho(Z) \subset \mathcal{V}_b(Z)$$

consists of those vector fields which additionally tangent to the fibers of ϱ . The further subspace of *gluing vector fields* is

$$\mathcal{V}_\gamma(Z) = \rho_D \rho_B \mathcal{V}_\varrho(Z).$$

In fact

$$\mathcal{V}_\gamma(Z) \subset \mathcal{V}_\varrho(Z) \subset \mathcal{V}_b(Z) \subset \mathcal{V}(Z) \quad (3.4)$$

are Lie subalgebras by an easy computation.

The vector bundles eTZ and ${}^\gamma TZ$ are defined by the property that

$$\mathcal{V}_\varrho(Z) = C^\infty(Z; {}^eTZ), \quad \mathcal{V}_\gamma(Z) = C^\infty(Z; {}^\gamma TZ).$$

The inclusions (3.4) induce bundle maps ${}^\gamma TZ \longrightarrow {}^eTZ$ and ${}^eTZ \longrightarrow {}^bTZ$.

Proposition 3.1. *There is a natural vector bundle isomorphism*

$$\pi_X^*({}^{sc}TX) \cong {}^\gamma TZ.$$

Proof. The subset $Z' = Z \setminus \varrho^{-1}(0)$ is canonically isomorphic to the product $X \times (0, \infty)$ and it is clear from the definitions that if $p \in Z'$, the fibre $({}^\gamma TZ)_p$ is therefore canonically isomorphic to ${}^{sc}T_{\pi_X(p)}X$. Thus we have a canonical isomorphism

$$\pi_X^*({}^{sc}TZ)|_{Z'} = ({}^\gamma TZ)|_{Z'} \quad (3.5)$$

and we need to show that this isomorphism extends over $\varrho^{-1}(0)$; this will be done in local coordinates.

As a matter of notation, if ξ is a section of $\text{pr}_1^*({}^{sc}TX)$ over a subset of $X \times (0, \infty)$, we denote by $\pi_X^*(\xi)$ the *lift* of ξ to Z , defined over Z' by (3.5) and then by extension by continuity to the boundary.

Consider first an interior point of $D \subset Z$. Near such a point, we have local coordinates (s, ω, ε) , the maps π_X and ϱ being

$$\pi_X(s, \omega, \varepsilon) = (\rho = \varepsilon s, \omega), \quad \varrho(s, \omega, \varepsilon) = \varepsilon,$$

where ρ is a defining function for ∂X in X and ω denotes some set of coordinates on $\partial X = \mathbb{S}^2$. One calculates

$$\pi_X^*(\rho \partial_\rho) = s \partial_s, \quad \pi_X^*(\partial_\omega) = \partial_\omega$$

from which it follows that the local frame $\{\rho^2 \partial_\rho, \rho \partial_\omega\}$ of ${}^{sc}TX$ lifts to the local frame $\{\varepsilon s^2 \partial_s, \varepsilon s \partial_\omega\}$ of ${}^\gamma TZ$. Hence (3.5) extends smoothly over $\text{Int}(D)$.

Next, consider an interior point p of one of the X_j . As in (3.3), we have local coordinates (v, ε) in a neighbourhood of p and by (3.2),

$$\pi_X(v, \varepsilon) = v + \frac{\zeta_j}{\varepsilon}, \quad \varrho(v, \varepsilon) = \varepsilon.$$

Thus for fixed ζ_j and ε , $\pi_X^* \partial_z = \partial_v$. Since (by construction of ${}^{sc}TX$) the euclidean vector fields ∂_z define a smooth frame of ${}^{sc}TX$ and the ∂_v define a smooth frame of ${}^\gamma TZ$ near X_j , we see that (3.5) extends smoothly over \mathring{X}_j .

Near the corner $D \cap X_j$, we have local coordinates (r, x, ω) , where r is the restriction of ρ_j and x is the restriction of ρ_D . In these coordinates,

$$\pi_X(r, x, \omega) = (\rho = x, \omega), \quad \varrho(r, x, \omega) = rx.$$

Thus

$$\pi_X^*(\rho \partial_\rho) = x \partial_x - r \partial_r, \quad \pi_X^*(\partial_\omega) = \partial_\omega.$$

Hence the local frame $\{\rho^2 \partial_\rho, \rho \partial_\omega\}$ lifts to give the local frame $\{x^2 \partial_x - rx \partial_r, x \partial_\omega\}$ of ${}^\gamma TZ$.

Finally, near $D \cap B$, we work in coordinates $(\varepsilon, x', \omega)$ where $x' = s^{-1} = \varepsilon/x$. Then

$$\pi_X^*(\{\rho^2 \partial_\rho, \rho \partial_\omega\}) = \{\varepsilon(x')^2 \partial_{x'}, \varepsilon x' \partial_\omega\},$$

and the identification extends also to points near $D \cap B$.

Note in particular that on global sections, we have a ‘lifting map’ π_X^*

$$\pi_X^*(\mathcal{V}_{\text{sc}}(X)) \subset \mathcal{V}_\gamma(Z). \quad (3.6)$$

□

Note that

$${}^\gamma TZ = \rho_D \rho_B {}^e TZ = \varepsilon \rho_X^{-1} \rho_B {}^e TZ, \quad (3.7)$$

where the bundle $\rho_X^{-1} \rho_B {}^e TZ$ can be defined (as for other rescaled versions of the tangent bundle) by its space of sections so that

$$\rho_X C^\infty(Z; \rho_X^{-1} \rho_B {}^e TZ) = \rho_B C^\infty(Z; {}^e TZ).$$

We observe that the restriction $\rho_X^{-1} \rho_B {}^e TZ|_D$ of this bundle is precisely the conic tangent bundle ${}^c TD$ introduced in §2.9.

The point of this is that the scaling map $\mu : {}^\gamma TZ \longrightarrow {}^\gamma TZ$ given by multiplication by ε , which vanishes over the boundary, extends to define a global isomorphism

$$\mu : \rho_B \rho_X^{-1} {}^e TZ \xrightarrow{\cong} {}^\gamma TZ.$$

Proposition 3.2. *There are natural vector bundle isomorphisms*

$${}^\gamma TZ|_{X_j} \cong {}^{\text{sc}} TX_j, \quad j = 0, \dots, N.$$

Composition of μ^{-1} and restriction to D determines a “rescaled restriction” isomorphism

$$\mu^{-1} : {}^\gamma TZ|_D \cong {}^c TD. \quad (3.8)$$

Proposition 3.3. *Let g be a scattering metric on X , and set $\tilde{g} = \pi_X^*(g)$. Then \tilde{g} is a smooth metric on ${}^\gamma TZ$ with the following properties:*

- (a) *The restriction g_0 of \tilde{g} to X_0 is identically equal to g .*
- (b) *The restriction g_j of \tilde{g} to X_j is a scattering metric, which is Euclidean with respect to the identification $X_i \cong \overline{\mathbb{R}^3}$.*
- (c) *With respect to (3.8), the rescaled restriction $g_D \cong g|_D$ defines a smooth cone metric on D , i.e., a positive definite section of ${}^c TD$. In fact, g_D is the lift over $D \longrightarrow \mathbb{S}^2 \times [0, \infty]$ of the Euclidean metric on $\overline{\mathbb{R}^3}$.*

- (d) *If g is the Euclidean metric on X to begin with, then over a product neighborhood of B , π_X^*g is independent of ε .*

Proof. Parts (a), (b), and (c) hold if X is a general scattering manifold with boundary Y and we prove it in this generality.

Note first that Proposition 3.1 implies that the lift \tilde{g} is a smooth metric (uniformly up to the boundary) on ${}^\gamma TZ$. It is thus enough to prove (a)—(d) at interior points of the boundary hypersurfaces of Z .

The hypersurface D_1 of Z_1 , being the (positive) projectivization of the normal bundle of the corner $Y \times \{0\}$ in Z_0 , is diffeomorphic to the cylinder $Y \times [0, \infty]$. Here we think of $[0, \infty]$ as the set of ratios $[\rho : \varepsilon]$ where ρ is a given defining function of the boundary hypersurface Y .

We calculate first the lift of an exact scattering metric g on X to Z_1 . If

$$g = \frac{d\rho^2}{\rho^4} + \frac{h(\rho, y; dy)}{\rho^2} \quad (3.9)$$

and $s = \varepsilon/\rho$, we have, near D ,

$$g = \frac{1}{\varepsilon^2} (g_D + \mathcal{O}(\varepsilon)) \quad (3.10)$$

where

$$g_D = ds^2 + s^2 h(0, y; dy) \quad (3.11)$$

and $\mathcal{O}(\varepsilon)$ denotes a section of $S^2 T^*$ whose length with respect to g_D is $\mathcal{O}(\varepsilon)$, uniformly if s is bounded away from 0. This proves part (c), at least away from $B \cap D_1$. Near this corner, we should use $\tilde{s} = \rho/\varepsilon$ and then the lift of the metric takes the form

$$g = \frac{1}{\varepsilon^2} \left(\frac{d\tilde{s}^2}{\tilde{s}^4} + \frac{h(\varepsilon\tilde{s}, y, dy)}{\tilde{s}^2} \right) \quad (3.12)$$

For the euclidean metric, we have that h is independent of ρ , and in this case

$$\varepsilon^2 g = \frac{d\tilde{s}^2}{\tilde{s}^4} + \frac{h(y, dy)}{\tilde{s}^2}$$

is independent of ε as required, proving part (d). The analogue of part (d) holds in the general setting of scattering manifolds if g is exactly conical in a collar neighbourhood of Y .

Now consider an interior point p of D_1 . If we choose local coordinates $(\zeta^\mu) = (\zeta^1, \dots, \zeta^n)$ in D_1 centred at p , then the lifted metric can be written

$$\tilde{g} = \frac{1}{\varepsilon^2} (\tilde{g}_{\mu\nu}(\zeta) d\zeta^\mu d\zeta^\nu + \mathcal{O}(\varepsilon))$$

Blowing up p corresponds to introducing $v^\mu = \zeta^\mu/\varepsilon$ as new coordinates, and the metric lifts as

$$\tilde{g}(\varepsilon v)_{\mu\nu} dv^\mu dv^\nu + \mathcal{O}(\varepsilon)$$

so that the restriction to the new face is the euclidean metric

$$\tilde{g}(p)_{\mu\nu} dv^\mu dv^\nu.$$

This proves (a) and (b). \square

3.3. Bogomolny equation. We begin with a general discussion of smooth bundles and connections over manifolds with corners such as Z : compare with the parallel discussion for manifolds with boundary in §2.1. In general if Z is a manifold with corners, a smooth principal G -bundle $P \rightarrow Z$ is defined in terms of smoothness of the data up to (and, as always, including) the boundary. It is equivalent to assume that $Z \subset \hat{Z}$ where \hat{Z} is some manifold without boundary of the same dimension as Z , and that P is the restriction to Z of some smooth G -bundle on \hat{Z} . A smooth G -connection A on P is defined similarly in terms of smoothness of data up to the boundary (which is again equivalent to the assumption that A is the restriction of a smooth connection on $\hat{P} \rightarrow \hat{Z}$).

Let V be any bundle associated to P . If H is any boundary hypersurface of Z , then there is a restriction map on sections of V ; also A defines a restricted connection A_H on $P|_H \rightarrow H$.

Conversely if $\{H_a\}$, $a = 1, \dots, p$, is an enumeration of the boundary hypersurfaces of Z and $s_a \in C^\infty(H_a, V|_{H_a})$, satisfying the compatibility conditions

$$s_a|_{H_a \cap H_b} = s_b|_{H_a \cap H_b} \quad (3.13)$$

for all pairs a and b , then there is an *extension* $s \in C^\infty(Z, V)$ of the s_a , i.e. $s|_{H_a} = s_a$ for each a . This follows from the (much more general) results in [Mel92]. The key point is that if s_a vanishes at $H_a \cap H_b$, then it has an extension \tilde{s}_a which vanishes to the same order at H_b . To see how this implies the general extension result, suppose by induction that we have already have a partial extension \tilde{s} with the property $\tilde{s}|_{H_a} = s_a$ for $a = 1, \dots, b-1$. Let $t = s_b - \tilde{s}|_{H_b}$. By the compatibility conditions (3.13),

$$t|_{H_a \cap H_b} = 0 \text{ for } a = 1, \dots, b-1.$$

Let \tilde{t} be an extension of t to Z which also vanishes at the H_a for $a = 1, \dots, b-1$. Then $\tilde{s} + \tilde{t}$ is smooth and its restriction to H_a is s_a for $a = 1, \dots, b$, completing the inductive step.

As in Proposition 2.3 there is an analogous result for connections: if (A_a) is a collection of connections in $P|_{H_a}$, which agree over $H_a \cap H_b$, then there is a connection A on $P \rightarrow Z$ such that $A|_{H_a} = A_a$ for each a .

With these preliminaries out of the way, suppose that $P = \pi_X^* P_X$ is a principal $SU(2)$ -bundle on our gluing space Z with a smooth connection A , $P_X \rightarrow X$ being a principal $SU(2)$ -bundle over X . Let $V \rightarrow Z$ be any associated vector bundle. By composing the covariant derivative operator

$$\hat{d}_A : C^\infty(Z; V) \rightarrow C^\infty(Z; T^*Z \otimes V)$$

with the natural map

$$T^*Z \rightarrow \gamma T^*Z \quad (3.14)$$

(dual to the inclusions (3.4)) we define the associated γ covariant derivative

$$d_A : C^\infty(Z; V) \rightarrow C^\infty(Z; \gamma T^*Z \otimes V).$$

This operator, which differentiates along the fibres of ϱ , will be our main concern, which is the reason for writing \widehat{d}_A for the ‘ordinary’ covariant derivative. We will be interested in the case $V = \mathfrak{p}$.

As a matter of notation, we write $\gamma\Lambda^k$ for the bundle $\bigwedge^k \gamma T^*Z$, and we note that the projection map (3.14) extends to define

$$C^\infty(Z; \Lambda^k) \longrightarrow (\rho_B \rho_D)^k C^\infty(Z; \gamma\Lambda^k)$$

since (3.14) factors through ${}^e T^*Z = (\rho_D \rho_X) \gamma T^*Z$. The following is immediate:

Lemma 3.4. *If A is a smooth connection and $\Phi \in C^\infty(Z; \mathfrak{p})$, then*

$$\begin{aligned} d_A \Phi &\in (\rho_B \rho_D) C^\infty(Z; \gamma\Lambda^1 \otimes \mathfrak{p}), \\ F_A &\in (\rho_B \rho_D)^2 C^\infty(Z; \gamma\Lambda^2 \otimes \mathfrak{p}). \end{aligned} \tag{3.15}$$

We consider now the fiberwise Bogomolny equation on Z :

$$\mathcal{B}(A, \Phi) = *F_A - d_A \Phi = 0, \tag{3.16}$$

where A is a (relative) connection, Φ is a section of \mathfrak{p} , and

$$* = *\tilde{g} : \gamma\Lambda^k \xrightarrow{\cong} \gamma\Lambda^{3-k}$$

is the relative Hodge star induced by the γ metric \tilde{g} .

As the fibers of Z for $\varepsilon > 0$ are canonically identified with (X, g) , with $\gamma T Z \cong {}^{\text{sc}} T X$ there, a solution to (3.16) represents a smooth family of Euclidean monopoles parameterized by $\varepsilon \in (0, \infty)$. We shall construct such families³ by extending monopole data initiall defined on $\varrho^{-1}(0)$ to nearby fibers $\varrho^{-1}(\varepsilon)$.

From Lemma 3.4 we obtain:

Proposition 3.5. *For any smooth data (A, Φ) on Z , we have*

$$\mathcal{B}(A, \Phi) = \rho_B \rho_D C^\infty(Z, \gamma\Lambda^1 \otimes \mathfrak{p}).$$

If the restriction (A_j, Φ_j) of (A, Φ) satisfies

$$\mathcal{B}(A_j, \Phi_j) = 0 \text{ on } X_j$$

for each j , then

$$\mathcal{B}(A, \Phi) \in \rho_B \rho_D \rho_X C^\infty(Z, \gamma\Lambda \otimes \mathfrak{p}).$$

Finally, if in addition $\nabla_{A|D}(\Phi|D) = 0$ over D ,

$$\mathcal{B}(A, \Phi) \in \rho_B^2 \rho_D^2 \rho_X C^\infty(Z, \gamma\Lambda \otimes \mathfrak{p}) \tag{3.17}$$

³Although they will only be conormal, not smooth

3.4. Ideal monopoles and pregluing configurations.

Definition 3.6. An *ideal* $SU(2)$ -monopole, ι , on Z , is the restriction to $\varrho^{-1}(0)$ of a smooth (A, Φ) defined near the boundary of Z , satisfying (3.17), and such that $A|_D$ is an $SU(2)$ Dirac monopole connection. In this situation we say that (A, Φ) *represents* the ideal monopole ι .

In the next section we shall discuss this notion further and in particular moduli spaces of ideal monopoles. It would be possible to give a definition which is intrinsic to $\varrho^{-1}(0)$ in terms of bundles and monopole data over D and the X_j which agree at all the corners, but (3.6) will serve our present purposes.

Proposition 3.7. *Given any collection of monopoles (A_j, Φ_j) , with*

$$[(A_0, \Phi_0)] \in \mathcal{M}_{k_0}, \quad [(A_j, \Phi_j)] \in \mathcal{M}_{k_j} \text{ for } j = 1, \dots, N,$$

there exists an ideal monopole whose restriction to X_j is the given data (A_j, Φ_j) .

Proof. Let $\mathcal{U} = \{0 \leq \rho_D < \delta\}$ be a product neighbourhood of D . It is convenient to work with vector bundles, so we denote by E the complex rank-2 vector bundle associated to P by the fundamental representation of $SU(2)$. We can equip the restriction $E|_{X_j}$ with the monopole data (A_j, Φ_j) for each j . We can also assume that δ is chosen so small $\Phi_j|_{\mathcal{U} \cap X_j}$ is non-zero. Then as in §2 we have the decomposition

$$E|_{\mathcal{U} \cap X_j} = L_j \oplus L_j^{-1} \tag{3.18}$$

into the eigenbundles of Φ_j . With respect to (3.18), we have

$$\Phi_j = \text{diag}(im - \phi_j, -im + \phi_j)$$

and

$$A_j = \text{diag}(a_j, -a_j) + b_j \tag{3.19}$$

where

$$\phi_j = \frac{ik_j}{2} \rho_D + \mathcal{O}(\rho_D^2) \tag{3.20}$$

and

$$b_j \in \rho_D^\infty C^\infty(\mathcal{U} \cap X_j, T^*X_j \otimes \mathfrak{p}_1).$$

In (3.19), we have committed a slight abuse of notation; it is to be understood that a_j is a connection in L_j and $-a_j$ is the dual connection in L_j^{-1} . By Proposition 2.8 we have

$$\dagger F(a_j) = \frac{ik_j}{2}. \tag{3.21}$$

Now let $L \longrightarrow \mathcal{U}$ be a complex line bundle with

$$L|_{\mathcal{U} \cap X_j} = L_j.$$

By (3.21), the data $\{a_j|_{D \cap X_j}\}$ comprise a Dirac framing in the sense of Definition 2.14. By Proposition 2.15, there exists a Dirac connection a_D

on L agreeing with a_j over $D \cap X_j$. Hence there exists a connection \tilde{a}_D on L over \mathcal{U} which agrees a_j over $X_j \cap \mathcal{U}$ and with a_D over D . We now define a connection A on $L \oplus L^{-1}$ by choosing smooth extensions \tilde{b}_j of b_j , $\tilde{b}_j \in \rho_D^\infty C^\infty(Z, {}^\gamma T^*Z \otimes \mathfrak{p}_1)$ and putting

$$A = \text{diag}(\tilde{a}, -\tilde{a}) + \sum_{j=0}^N \tilde{b}_j.$$

Then A is smooth and it restricts to A_j over X_j and to the Dirac connection A_D over D .

If we define $\Phi_D = \text{diag}(im, -im)$ with respect to the decomposition $E|_D = L \oplus L^{-1}$, then $\nabla_{A|D} \Phi_D = 0$ and moreover Φ_D agrees over $D \cap X_j$ with Φ_j for each j . Hence there is a smooth extension $\Phi \in C^\infty(Z; \text{ad}(P))$ of these data over Z . Then (A, Φ) represents the given ideal monopole configuration as required. \square

Before starting on solving the fibrewise Bogomonly equations in earnest, it is convenient to introduce a better choice of smooth configuration representing a given ideal monopole. Thus we make the following definition:

Definition 3.8. A *pregluing configuration* is a smooth configuration (A, Φ) on Z with the following properties:

(i)

$$\mathcal{B}(A, \Phi) = \mathcal{O}(\rho_X \rho_D^3 \rho_B^\infty) \quad (3.22)$$

(ii) ∇_A is diagonal to infinite order with respect to the splitting

$$\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_1 := \text{span}_{\mathbb{C}} \Phi \oplus \Phi^\perp \quad (3.23)$$

in a product neighbourhood of D .

Proposition 3.9. Let ι be an ideal monopole configuration on Z . Then there exists a pregluing configuration on Z which represents ι .

Proof. The proof is a continuation of the previous proposition and we make use of the notation established there. Let (A, Φ) be as constructed from the boundary data in the proof of Proposition 3.7. In particular, $\mathcal{B}(A, \Phi) = \mathcal{O}(\rho_B^2 \rho_D^2 \rho_X)$ and $A|_D$ is an $\text{SU}(2)$ Dirac connection.

To see how to improve the order of vanishing at D , use $\varepsilon = \rho_D \rho_X$ to write the Taylor expansion of Φ in \mathcal{U} in the form

$$\Phi = \Phi_0 + \varepsilon \Phi_1 + \mathcal{O}(\rho_D^2), \quad \Phi_1 \in \rho_X^{-1} C^\infty(D; \mathfrak{p}).$$

Then

$$d_A \Phi = \underbrace{d_A \Phi_0}_{\mathcal{O}(\rho_D)} + \underbrace{\varepsilon d_A \Phi_1}_{\mathcal{O}(\rho_D^2)} + \mathcal{O}(\rho_D^3)$$

as a section of ${}^\gamma \Lambda^1 \otimes \mathfrak{p}$. (Note that ε commutes with the γ connection d_A since $[\varepsilon, \mathcal{V}_\varrho] = 0$ by definition. By contrast, ρ_D does not commute with d_A .)

Hence

$$\mathcal{B}(A, \Phi) = -\underbrace{d_A \Phi_0}_{\mathcal{O}(\rho_D)} + \underbrace{*F_A - \varepsilon d_A \Phi_1}_{\mathcal{O}(\rho_D^2)} + \mathcal{O}(\rho_D^3) \in C^\infty(U; \gamma \Lambda^1 \otimes \mathfrak{p})$$

To interpret this in terms of the geometry of D we make use of the rescaled restriction isomorphism

$$(\mu^{-1})^* : \gamma \Lambda^k Z|_D \cong {}^c \Lambda^k D \quad (3.24)$$

dual to (3.8). There is a commutative diagram

$$\begin{array}{ccc} \gamma \Lambda^k Z|_D & \xrightarrow{d} & \gamma \Lambda^{k+1} Z|_D \\ \downarrow (\mu^{-1})^* & & \downarrow (\mu^{-1})^* \\ {}^c \Lambda^k D & \xrightarrow{\varepsilon d} & {}^c \Lambda^{k+1} D \end{array} \quad (3.25)$$

and likewise d_A is intertwined with $\varepsilon d_{A|D}$, where $d_{A|D}$ is the covariant derivative of scattering/conic type on D induced by the restriction of A . It follows that

$$\mathcal{B}(A, \Phi) = -\varepsilon d_{A|D} \Phi_0 + \varepsilon^2 (*_D F_{A|D} - d_{A|D} \Phi_1) + \mathcal{O}(\rho_D^3), \quad (3.26)$$

where we are regarding the coefficients of ε^k as sections of ${}^c \Lambda^1 D \otimes \mathfrak{p}$. Vanishing of the first term has already been discussed. In order for the term in parentheses to vanish, we need

$$*_D F_{A|D} - d_{A|D} \Phi_1 = 0 \quad (3.27)$$

which is to say that $A|D$ is an $SU(2)$ Dirac connection (cf. §2.9.1) with abelian Higgs field Φ_1 .

In the previous proposition, we constructed A so that $A|D$ was an $SU(2)$ Dirac monopole. So it remains to show that Φ can be constructed so that Φ_1 is an abelian Higgs field. For this, define

$$\phi_D = \frac{i}{2} \sum_{j=0}^N \frac{k_j}{|\zeta - \zeta_j|} \quad (3.28)$$

so that $*_D d\phi_D$ is the curvature $F(a_D)$ of a_D . If we choose $\rho_j = \rho_{X_j} = |\zeta - \zeta_j|$ near $D \cap X_j$, then clearly

$$\psi := \rho_0 \cdots \rho_N \phi_D \in C^\infty(D)$$

and $\psi|_{D \cap X_j} = ik_j/2$. Similarly $\psi_j = \rho_D^{-1} \phi_j$ is smooth on X_j and $\psi_j|_{D \cap X_j} = ik_j/2$ by (3.20). Let $\tilde{\psi}$ be a smooth extension of ψ_D and the ψ_j to Z , and let $\tilde{\phi} = (\rho_0 \cdots \rho_N)^{-1} \tilde{\psi}$. By construction, $\varepsilon \tilde{\phi}|_{X_j \cap \mathcal{U}} = \phi_j$, and near any interior point of D ,

$$\tilde{\phi} = \phi_D + \mathcal{O}(\varepsilon). \quad (3.29)$$

We now define $\Phi_1 = \text{diag}(\tilde{\phi}, -\tilde{\phi})$ and by construction $\Phi_0 + \varepsilon \Phi_1$ is smooth and has the correct Taylor expansion at D .

Finally, to arrange the vanishing of $\mathcal{B}(A, \Phi)$ to infinite order near B , note that a neighbourhood $U = \mathcal{U} \cap \{0 \leq \rho_B \leq \delta'\}$ can be identified with the product of $\{0 \leq \rho_D < \delta\}$ with a neighbourhood U' of S_∞ in D , in such a way that the fibrewise metric \tilde{g} on $\rho_D = \varepsilon$ is independent of ε in U (part (d) of Proposition 3.3). In such a neighbourhood we can choose the extensions \tilde{a} and $\tilde{\phi}$ also to be independent of ε . We may also suppose that all $\tilde{b}_j = 0$ in this neighbourhood. Then the argument leading to (3.26) has no error term here and gives

$$\mathcal{B}(A, \Phi) = *F_A - d_A\Phi \cong \varepsilon^2 *_{D|_D} F_{A|_D} - \varepsilon d_{A|_D}(\varepsilon\phi_D) \equiv 0 \text{ over } U$$

as required. \square

3.5. The iteration. Suppose (A, Φ) is a pregluing configuration. The task is now to find a perturbation (a, ϕ) such that $\mathcal{B}(A + a, \Phi + \phi)$ vanishes to high order in ε .

For $(a, \phi) \in C^\infty(Z; (\gamma\Lambda^1 \oplus \gamma\Lambda^0) \otimes \mathfrak{p})$,

$$\mathcal{B}(A + a, \Phi + \phi) = \mathcal{B}(A, \Phi) + D\mathcal{B}_{A, \Phi}(a, \phi) + Q(a, \phi)$$

$$D\mathcal{B}_{A, \Phi}(a, \phi) = *d_A a - d_A \phi + [\Phi, a],$$

$$Q(a, \phi) = *[a, a] - [a, \phi]$$

The linear operator $D\mathcal{B}_{A, \Phi}$ is not elliptic owing to the action of the gauge group. To remedy this, we impose the *Coulomb gauge* condition

$$d_{A, \Phi}^*(a, \phi) = d_A^* a - [\Phi, \phi] = 0 \in C^\infty(Z; \mathfrak{p}). \quad (3.30)$$

just as in the discussion in §2.6, where now d_A^* is the formal adjoint of the γ -covariant derivative d_A on \mathfrak{p} . Over the $\varepsilon > 0$ fibers of Z , this condition is known to determine a slice for the action of the gauge group provided $(a, \phi)|_\varepsilon$ is sufficiently small.

Later, when we take into account the parameters of the gluing construction, we will show that (3.30) determines a slice globally and uniformly in the parameters, for sufficiently small ε (c.f. Theorem 5.8).

Thus we seek to solve

$$\mathcal{B}(A, \Phi) + L(a, \phi) + Q(a, \phi) = 0$$

$$L = D\mathcal{B}_{A, \Phi} + d_{A, \Phi}^*, \quad (3.31)$$

$$L(a, \phi) = \begin{bmatrix} *d_A & -d_A \\ -d_A^* & 0 \end{bmatrix} \begin{bmatrix} a \\ \phi \end{bmatrix} + \text{ad}(\Phi) \begin{bmatrix} a \\ \phi \end{bmatrix}$$

where L is the operator (2.47) acting along the fibres of ϱ .

Over X_j , d_A restricts to the covariant derivative d_{A_j} acting on sections of ${}^{\text{sc}}\Lambda^* X_j \otimes \mathfrak{p} \cong \gamma\Lambda^* \otimes \mathfrak{p}|_{X_j}$, and $*$ is identified with the Hodge star associated to the metric g_j . A consequence is the following:

Proposition 3.10. *Denote by L_j the operator $L_{(A_j, \Phi_j)}$ over X_j . Then $L|_{X_j} = L_j$ in the sense that if $(a, \phi) \in C^\infty(Z; \gamma\Lambda \otimes \mathfrak{p})$, then*

$$(L(a, \phi))|_{X_j} = L_j((a, \phi)|_{X_j}) \in C^\infty(X_j; {}^{\text{sc}}\Lambda \otimes \mathfrak{p}).$$

At D the situation is a bit more complicated. First, we decompose L according to the splitting $\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_1$ near D . Note that, as a section of \mathfrak{p}_0 , Φ acts trivially on \mathfrak{p}_0 and nondegenerately on \mathfrak{p}_1 . Combined with the assumption that A is diagonal to infinite order, we may write

$$L = \begin{pmatrix} L_0 & \mathcal{O}(\rho_D^\infty \rho_B^\infty) \\ \mathcal{O}(\rho_D^\infty \rho_B^\infty) & L_1 + \Phi_1 \end{pmatrix} \quad (3.32)$$

with respect to $\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_1$, where Φ_1 denotes the nondegenerate restriction to \mathfrak{p}_1 .

We focus attention on L_0 ; we will not need to consider $L_1 + \Phi_1$ until §5. Using the fact that A is trivial on $\mathfrak{p}_0 = \mathfrak{q}$ at D , the diagram (3.25), and the fact that $*$ is intertwined with $*_D$ under the rescaled restriction isomorphism, we have the following

Proposition 3.11. *The rescaled restriction $\lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} L_0$ of L_0 is the operator*

$$L_D : C^\infty(D; {}^c\Lambda) \longrightarrow C^\infty(D; {}^c\Lambda),$$

$$L_D(a, \phi) = \begin{bmatrix} *_D d & -d \\ d^* & 0 \end{bmatrix} \begin{bmatrix} a \\ \phi \end{bmatrix}$$

in the following sense: if $(a, \phi) \in C^\infty(U; (\gamma\Lambda^1 \oplus \gamma\Lambda^0) \otimes \mathfrak{p}_0)$, then with respect to the rescaled restriction isomorphism (3.24)

$$(\varepsilon^{-1} L_0(a, \phi))|_D \cong L_D((a, \phi)|_D) \in C^\infty(D; {}^c\Lambda^1 \oplus {}^c\Lambda^0).$$

The operators L_X and L_D are analyzed in Appendix C, in particular the regularity of solutions to $L_\bullet u = f$ is discussed. In order to state these results, and for the formal construction below, we first need to introduce a weakening of smoothness to allow functions with asymptotic expansions having logs and noninteger powers of boundary defining functions.

3.6. Polyhomogeneity. Briefly, a polyhomogeneous function is one having a complete asymptotic expansion at all boundaries with terms of the form $x^s(\log x)^p$ where x is a boundary defining function. The exponents $(s, p) \in \mathbb{R} \times \mathbb{N}_0$ which appear in the expansion are recorded by a (real) *index set*, which is a discrete subset $E \subset \mathbb{R} \times \mathbb{N}_0$ satisfying the property that for each $\alpha \in \mathbb{R}$, there are only finitely many $(s, p) \in E$ with $s \leq \alpha$, and only finitely many (s, p) for each fixed s . An index set is *smooth* if

$$(s, p) \in E \implies (s + n, q) \in E, \quad \forall n \in \mathbb{N}_0, \quad 0 \leq q \leq p.$$

Unless otherwise specified, all index sets will be smooth.

Definition 3.12. If X is a manifold with boundary, then the polyhomogeneous space, $\mathcal{A}^E(X)$, is the space of smooth functions on $X \setminus \partial X$ having an

asymptotic expansion

$$u \sim \sum_{(s,p) \in E} u_{(s,p)} x^s (\log x)^p, \quad u_{(s,p)} \in C^\infty(\partial X)$$

where x is a boundary defining function for ∂X , and the equivalence is modulo $x^\infty C^\infty(X)$. The finiteness condition on E insures that such sums are Borel summable. If E is smooth then $\mathcal{A}^E(X)$ is independent of the choice of x .

Suppose now that X is a manifold with corners with boundary hypersurfaces H_1, \dots, H_N , and let $\mathcal{E} = (E_1, \dots, E_N)$ be an *index family*, meaning an N -tuple of smooth index sets. Then $\mathcal{A}^\mathcal{E}(X)$ is recursively defined as the space of smooth functions on $X \setminus \partial X$ having asymptotic expansions

$$u \sim \sum_{(s,p) \in E_k} u_{(s,p)} \rho_k^s (\log \rho_k)^p, \quad u_{(s,p)} \in \mathcal{A}^{\mathcal{E}(k)}(H_k).$$

Here $\mathcal{E}(k)$ is the index family obtained from \mathcal{E} which is associated to boundary hypersurfaces of H_k , which consist of connected components of $H_k \cap H_j$ for various j .

If $V \longrightarrow X$ is a vector bundle, the spaces $\mathcal{A}^\mathcal{E}(X; V)$ are defined as above in terms of local trivializations.

There are a number of notational conventions and operations on index sets which we use below. First, we identify $s \in \mathbb{R}$ with the smallest smooth index set containing $(s, 0)$, namely $\{(s+n, 0) : n \in \mathbb{N}_0\}$. In particular, $\mathcal{A}^s(X) \equiv x^s C^\infty(X)$ for a manifold with boundary, and $\mathcal{A}^{(s_1, \dots, s_N)}(X) = \rho_1^{s_1} \dots \rho_N^{s_N} C^\infty(X)$ for a manifold with corners. We also write ∞ for the empty index set, which is consistent with the identity $\mathcal{A}^\infty(X) = x^\infty C^\infty(X)$.

We order elements of $\mathbb{R} \times \mathbb{N}_0$ lexicographically, with the opposite order on \mathbb{N}_0 , so that

$$(s, p) < (t, q) \iff s < t, \text{ or } s = t \text{ and } p > q.$$

(This is consistent with the idea that $\mathcal{O}(x^s (\log x)^p)$ has worse decay than $\mathcal{O}(x^t (\log x)^q)$ as $x \rightarrow 0$.) We then order index sets by comparing their minimal elements (which exist by the finiteness conditions) with respect to this order: thus $E < F$ (respectively $E \leq F$) if and only if $\min E < \min F$ (respectively $\min E \leq \min F$). In particular, for $m \in \mathbb{R}$,

$$\begin{aligned} E > m &\iff \min \{s : (s, p) \in E\} > m, \\ E \geq m &\iff E > m, \text{ or } (m, 0) \in E \text{ but } (m, p) \notin E \ \forall p \geq 1, \end{aligned}$$

and if X is a manifold with boundary, $\mathcal{A}^E(X) \subset C^0(X)$ if and only if $E \geq 0$ and $\mathcal{A}^E(X) \subset C_0^0(X)$ if and only if $E > 0$.

If E and F are (smooth) index sets, then so too are $E + F$ and $E \cup F$. These correspond respectively to multiplication and addition of sections of

V , in the sense that

$$\begin{aligned}\mathcal{A}^{\mathcal{E}}(X; V) \times \mathcal{A}^{\mathcal{F}}(X; W) &\ni (u, v) \longmapsto u \otimes v \in \mathcal{A}^{\mathcal{E}+\mathcal{F}}(X; V \otimes W), \\ \mathcal{A}^{\mathcal{E}}(X; V) \times \mathcal{A}^{\mathcal{F}}(X; V) &\ni (u, v) \longmapsto u + v \in \mathcal{A}^{\mathcal{E} \cup \mathcal{F}}(X; V).\end{aligned}$$

The *extended union* of E and F , denoted $E \overline{\cup} F$, is the index set

$$E \overline{\cup} F = E \cup F \cup \{(s, p+q+1) : (s, p) \in E, (s, q) \in F\}.$$

This arises in the context of fiber integration [Mel92].

Finally, we introduce one more notational convention which will be used below. For $n \in \mathbb{Z}$, $k \in \mathbb{N}_0$, let

$$\overline{(n, k)} = \{(n+l, j) : l \in \mathbb{N}_0, 0 \leq j \leq k+l\}.$$

This is the smallest smooth index set containing $\{(n+l, k+l) : l \in \mathbb{N}_0\}$, and $\mathcal{A}^{\overline{(n, l)}}(X)$ consists of functions whose asymptotic expansions have logarithmic terms with powers growing linearly with the powers of x .

3.7. Formal solution. The key solvability properties of L_X and L_D , proved in Appendix C, are summarized in the following:

Theorem 3.13 (Thm. C.5.(b), Thm C.9.(b)).

- (a) Let $f \in \mathcal{A}^*(X; (\text{sc}\Lambda^1 \oplus \text{sc}\Lambda^0) \otimes \mathfrak{p})$ with $f = f_0 \oplus f_1$ near ∂X with $f_i \in \mathcal{A}^{F_i}(X; \text{sc}\Lambda^* \otimes \mathfrak{p}_i)$, $i = 1, 2$, and suppose $F_i > \frac{3}{2}$. Then there is a unique solution to $L_X u = f$ with u in $\text{Null}(L_X)^\perp$ with respect to the L^2 pairing over X , and $u \in \mathcal{A}^*(X; (\text{sc}\Lambda^1 \oplus \text{sc}\Lambda^0) \otimes \mathfrak{p})$ with $u = u_0 \oplus u_1 \in \mathcal{A}^{E_0} \oplus \mathcal{A}^{E_1}$ near ∂X , where

$$E_0 = \overline{(2, 0)} \overline{\cup} (F_0 - 1), \quad E_1 = F_1.$$

- (b) let $f \in \mathcal{A}^{F_X, F_B}(D; {}^c\Lambda^1 \oplus {}^c\Lambda^0)$ with $F_B > \frac{3}{2}$ and $F_X > -\frac{3}{2}$. Then there exists a unique solution to $L_D u = f$ with $u \in \mathcal{A}^{E_X, E_B}(D; {}^c\Lambda^1 \oplus {}^c\Lambda^0)$ where

$$E_X = \overline{(0, 0)} \overline{\cup} (F_X + 1), \quad E_B = \overline{(2, 0)} \overline{\cup} (F_B - 1),$$

This leads to the following technical result, which is fundamental to the iterative step in our construction.

Lemma 3.14. Denote the bundle $(\gamma\Lambda^1 \oplus \gamma\Lambda^0) \otimes \mathfrak{p}$ by W , and in a neighborhood U of $D \cup B$, write $W_i = (\gamma\Lambda^1 \oplus \gamma\Lambda^0) \otimes \mathfrak{p}_i$. In the following statements, both f and u have rapidly vanishing W_1 components in U , i.e., the W_1 projections of their restrictions to U lie in $\mathcal{A}^{\infty_{D, \infty_B, *X}}(U; W_1)$.

- (a) Let $f \in \mathcal{A}^{\overline{(2, j)}_D, 0_X, \infty_B}(Z; W)$. Then there exists $u \in \mathcal{A}^{\overline{(1, j+1)}_D, 0_X, \infty_B}(Z; W)$ such that

$$Lu - f \in \mathcal{A}^{\overline{(2, j+1)}_D, 1_X, \infty_B}(Z; W).$$

- (b) Let $f \in \mathcal{A}^{0_D, \overline{(-1, j)}}_{X, \infty_B}(Z; W)$ be arbitrary. Then there exists $u \in \mathcal{A}^{0_D, \overline{(0, j+1)}}_{X, \overline{(2, 0)}_B}(Z; W)$ supported in a neighborhood of D , such that

$$Lu - \varepsilon f \in \mathcal{A}^{2_D, \overline{(0, j+1)}}_{X, \infty_B}(Z; W),$$

and furthermore, the W_1 projection of u vanishes identically.

Observe that the index set $0 = (0, 0)$ means a smooth expansion up to a given face, 1 and 2 means smooth and vanishing to first and second order, respectively, and ∞ means rapid decay.

Proof. By Theorem 3.13, we can solve $L_X u_X = f|_X$ for a unique $u_X \in \mathcal{A}^E(X) \cap \text{Null}(L_X)^\perp$ where

$$\begin{aligned} E_0 &= \overline{(2, 0)} \cup \overline{(1, j)} \subset \overline{(1, j+1)}, \\ E_1 &= F_1 = \infty. \end{aligned} \tag{3.33}$$

In a product neighborhood of X which does not meet B , we define

$$u = \chi(\rho_X) u_X \in \mathcal{A}^{\overline{(1, j+1)}}_{D, 0_X, \infty_B}(Z; W)$$

where χ is a smooth cutoff and we confuse u_X with its pullback to a function independent of ρ_X . Now L can be expressed as $\rho_D \rho_B$ times a first order operator of ϱ type; in particular we have

$$L : \mathcal{A}^{F_D, F_X, F_B}(Z; W) \longrightarrow \rho_D \rho_B \mathcal{A}^{F_D, F_X, F_B}(Z; W) = \mathcal{A}^{F_D+1, F_X, F_B+1}(Z; W),$$

so it follows that $Lu \in \mathcal{A}^{\overline{(2, j+1)}}_{D, 0_X, \infty_B}$. However, by construction Lu and f are polyhomogeneous sections smooth in ρ_X which have the same restriction to X , so their difference vanishes to first order there.

The second result is similar. We solve $L_D u_D = f|_D$ for a unique $u_D \in \mathcal{A}^{E_X, E_B}(D; W_0)$ with

$$E_X = \overline{(0, 0)} \cup \overline{(0, j)} = \overline{(0, j+1)}, \quad E_B = \overline{(2, 0)}.$$

Letting u be any smooth extension of u_D off of D as a section of W with identically vanishing \mathfrak{p}_1 component, it follows that $L(\varepsilon^{-1}u) = \varepsilon^{-1}L(u) \in \mathcal{A}^{0_D, \overline{(-1, j+1)}}_{X, \overline{(3, 0)}_B}$ and f have the same restriction at D , so

$$Lu - \varepsilon f \in \varepsilon \mathcal{A}^{1_D, \overline{(-1, j+1)}}_{X, \overline{(3, 0)}_B}(Z; W) = \mathcal{A}^{2_D, \overline{(0, j+1)}}_{X, \overline{(3, 0)}_B}(Z; W).$$

Finally, since there is a neighborhood of B in which A and the metric are independent of ε , Proposition 3.11 extends to say that the restriction of L_0 to any ε fiber of this neighborhood agrees with εL_D there, so we may take the extension u over this neighborhood to satisfy $\varepsilon L_D u_0(\varepsilon) = \varepsilon f_0(\varepsilon)$, where f_0 is the W_0 projection of f , and then it follows that

$$Lu - \varepsilon f \in \mathcal{A}^{2_D, \overline{(0, j+1)}}_{X, \infty_B}(Z; W)$$

since $Lu_0 - \varepsilon f_0$ vanishes identically near B while u_1 and f_1 are rapidly vanishing. \square

We return now to the equation (3.31), and the main result of this section.

Theorem 3.15. *Let (A, Φ) be a smooth pregluing configuration as in Definition 3.8. Then there exists a solution $(a, \phi) \in \mathcal{A}^{\mathcal{F}}(Z; (\gamma\Lambda^1 \oplus \gamma\Lambda^0) \otimes \mathfrak{p})$ to*

$$\mathcal{B}(A, \Phi) + L(a, \phi) + Q(a, \phi) = 0 \mod \varepsilon^\infty C^\infty(Z; (\gamma\Lambda^1 \oplus \gamma\Lambda^0) \otimes \mathfrak{p})$$

where

$$\begin{aligned} \mathcal{F} &= (F_D, F_X, F_B), \quad F_D = \{(n, 2n-4) : 2 \leq n \in \mathbb{N}\}, \\ F_X &= (1, 0) \cup \{(n, 2n-3) : 2 \leq n \in \mathbb{N}\}, \quad F_B = \overline{(2, 0)}. \end{aligned} \quad (3.34)$$

In other words, $\mathcal{B}(A+a, \Phi+\phi) = 0$ (along with the Coulomb gauge condition $d_{(A, \Phi)}^*(a, \phi) = 0$) is satisfied up to an error which is smooth on Z and rapidly vanishing in ε .

Furthermore, the \mathfrak{p}_1 components of (a, ϕ) are rapidly vanishing in ρ_D and ρ_B , and each coefficient in the expansion of (a, ϕ) at X_j is L^2 orthogonal to $\text{Null}(L_X)$.

Proof. Define

$$N(a, \phi) = L(a, \phi) + Q(a, \phi) + \mathcal{B}(A, \Phi),$$

so we wish to solve $N(a, \phi) = 0 \mod \varepsilon^\infty$. For notational convenience, for the remainder of the section we will denote (a, ϕ) by a single letter and omit the bundle $(\gamma\Lambda^1 \oplus \gamma\Lambda^0) \otimes \mathfrak{p}$ from the notation.

Though the proof is by induction, we take the first few steps by hand to illustrate the main idea before stating the full inductive step. From the construction of the initial data (A, Φ) , we begin with

$$N(0) = \mathcal{B}(A, \Phi) = \varepsilon f, \quad f \in \mathcal{A}^{2D, 0X, \infty B}(Z),$$

with \mathfrak{p}_1 components which are rapidly vanishing in ρ_D and ρ_B . Expanding in ρ_X , We write the error as

$$N(0) = \varepsilon(f' + f''), \quad f' \in \mathcal{A}^{2D, 0X, \infty B}(Z), \quad f'' \in \mathcal{A}^{2D, 1X, \infty B}(Z).$$

By Lemma 3.14, there exists $v \in \mathcal{A}^{\overline{(1,0)}_{D, 0X, \infty B}}(Z)$ such that $Lv + f' \in \mathcal{A}^{\overline{(2,0)}_{D, 1X, \infty B}}(Z)$ (that the index set is $\overline{(2,0)}$ rather than $\overline{(2,1)}$ at D follows from the fact that the forcing term is smooth; in (3.33) we have $\overline{(2,0)} \cup (1,0) \subset \overline{(1,0)}$). Since v has rapidly vanishing \mathfrak{p}_1 components at $D \cup B$, and since the quadratic pairing is trivial on \mathfrak{p}_0 , it follows that $Q(v) \in \mathcal{A}^{\infty D, 0X, \infty B}(Z)$. Then

$$\varepsilon v \in \mathcal{A}^{\mathcal{F}}(Z),$$

$$N(\varepsilon v) = \varepsilon(Lv + f') + Q(\varepsilon v) + \varepsilon f'' = \varepsilon \tilde{h}, \quad \tilde{h} \in \mathcal{A}^{\overline{(2,0)}_{D, 1X, \infty B}}(Z).$$

(Here we have used that L commutes with ε and $Q(\varepsilon v) = \varepsilon^2 Q(v)$.) Since v has rapidly vanishing \mathfrak{p}_1 components, so does this new error term.

The next step is to factor out $\varepsilon^2 = (\rho_D \rho_X)^2$ and, expanding in ρ_D , write the new error term as

$$N(\varepsilon v) = \varepsilon^3(h' + h''), \quad h' \in \mathcal{A}^{0D, -1X, \infty B}(Z), \quad h'' \in \mathcal{A}^{\overline{(1,1)}_{D, -1X, \infty B}}(Z).$$

There exists $w \in \mathcal{A}^{0D, \overline{(0,1)}_X, \overline{(2,0)}_B}(Z)$ such that $Lw + \varepsilon h' \in \mathcal{A}^{2D, \overline{(0,1)}_X, \infty_B}(Z)$ by Lemma 3.14, and then

$$\begin{aligned} \varepsilon^2 w &\in \mathcal{A}^{\mathcal{F}}(Z), \\ N(\varepsilon v + \varepsilon^2 w) &= \varepsilon^2(\tilde{f} + \tilde{g}), \\ \tilde{f} = \varepsilon h'' &\in \mathcal{A}^{(2,1)D, 0X, \infty_B}(Z) \\ \tilde{g} &= (Lw + \varepsilon h') + \varepsilon 2Q(v, w) + \varepsilon^2 Q(w) = Lw + \varepsilon h' \in \mathcal{A}^{2D, \overline{(0,1)}_X, \infty_B}(Z). \end{aligned}$$

Here we abuse notation by confusing Q and its associated bilinear form, and we use the fact that since w is supported near D and has identically vanishing \mathfrak{p}_1 component, $Q(w, \cdot) \equiv 0$. Again w , \tilde{f} and \tilde{g} have rapidly vanishing \mathfrak{p}_1 components.

A key point in the iteration to follow is that we keep separate track of the error terms which have growth in their powers of $\log \rho_D$ but not $\log \rho_X$, such as \tilde{f} above, and those which have growth in their powers of $\log \rho_X$ but not $\log \rho_D$, such as \tilde{g} .

Finally, observe that \tilde{g} has leading order at X given by $\log \rho_X$. Using the identity $\log \rho_X = \log \varepsilon - \log \rho_D$, this may be effectively removed, allowing us to write

$$\tilde{g} = \log \varepsilon g_0 + g_1, \quad g_j \in \mathcal{A}^{(2,j)D, \overline{(0,0)}_X, \infty_B}(Z),$$

where (n, m) denotes the smallest smooth index set containing (n, m) , namely $\{(k, l) : n \leq k, 0 \leq l \leq m\}$.

Now we begin the induction. Suppose that we have $u_n \in \mathcal{A}^{\mathcal{F}}(Z)$ with

$$\begin{aligned} N(u_n) &= \varepsilon^n (\log \varepsilon)^{2n-3} (g_0) + \varepsilon^n (\log \varepsilon)^{2n-4} (g_1 + f_1) + \\ &\quad \cdots + \varepsilon^n (\log \varepsilon)^0 (g_{2n-3} + f_{2n-3}) + \mathcal{O}(\varepsilon^{n+1}) \\ f_j &\in \mathcal{A}^{\overline{(2,j)}_D, 0X, \infty_B}(Z), \quad g_j \in \mathcal{A}^{(2,j)D, \overline{(0,0)}_X, \infty_B}(Z), \end{aligned}$$

all having rapidly vanishing \mathfrak{p}_1 components near $D \cup B$, and where $\mathcal{O}(\varepsilon^{n+1})$ denotes a finite number of terms of the form above with n replaced by $m > n$. Furthermore, suppose that, with respect to pairing by the bilinear form,

$$Q(u_n, \cdot) = \sum_{m=1}^{n-1} \varepsilon^m \sum_{j=0}^{2m-3} (\log \varepsilon)^j Q(\tilde{u}_{m,j}, \cdot), \quad \tilde{u}_{m,j} \in \mathcal{A}^{\infty_D, 0X, \infty_B}(Z), \quad (3.35)$$

i.e., with respect to multiplication, u_n has an expansion in $\varepsilon^m (\log \varepsilon)^j$ with coefficients which are smooth at X and rapidly vanishing elsewhere. The case $n = 2$ is furnished by $u_2 = \varepsilon v + \varepsilon^2 w$ above, with $f_1 = \tilde{f}$.

Expanding in ρ_X , we write

$$\begin{aligned} g_j &= g'_j + g''_j, \quad g'_j \in \mathcal{A}^{(2,j)D, 0X, \infty_B}(Z), \quad g''_j \in \mathcal{A}^{(2,j)D, \overline{(1,1)}_X, \infty_B}(Z), \\ f_j &= f'_j + f''_j, \quad f'_j \in \mathcal{A}^{\overline{(2,j)}_D, 0X, \infty_B}(Z), \quad f''_j \in \mathcal{A}^{\overline{(2,j)}_D, 1X, \infty_B}(Z). \end{aligned}$$

Invoking Lemma 3.14, for each j there exists $v_j \in \mathcal{A}^{\overline{(1,j+1)}_D, 0_X, \infty_B}(Z)$ such that $Lv_j + (g'_j + f'_j) \in \mathcal{A}^{\overline{(2,j+1)}_D, 1_X, \infty_B}(Z)$ for each j . Then

$$\begin{aligned} \varepsilon^n v &:= \varepsilon^n (\log \varepsilon)^{2n-3} v_0 + \cdots + \varepsilon^n (\log \varepsilon)^0 v_{2n-3} \in \mathcal{A}^{\mathcal{F}}(Z), \\ N(u_n + \varepsilon^n v) &= \varepsilon^n (\log \varepsilon)^{2n-3} (\tilde{h}_1 + \tilde{k}_0) + \varepsilon^n (\log \varepsilon)^{2n-4} (\tilde{h}_2 + \tilde{k}_1) + \\ &\quad \cdots + \varepsilon^n (\log \varepsilon)^0 (\tilde{h}_{2n-2} + \tilde{k}_{2n-3}) + R_{n+1} \\ \tilde{k}_j &= g''_j \in \mathcal{A}^{(2,j)_D, \overline{(1,1)}_X, \infty_B}(Z), \\ \tilde{h}_j &= (Lv_{j-1} + g'_{j-1} + f'_{j-1}) + f''_j \in \mathcal{A}^{\overline{(2,j)}_D, 1_X, \infty_B}(Z), \\ R_{n+1} &= 2Q(u_n, \varepsilon^n v) + Q(\varepsilon^n v) = \mathcal{O}(\varepsilon^{n+1}) \end{aligned}$$

Next, before solving at D , we factor out $\varepsilon^2 = (\rho_D \rho_X)^2$ and use the identity $(\log \rho_D)^j = (\log \varepsilon - \log \rho_X)^j$ to remove the leading powers of $\log \rho_D$ and distribute them as powers of $\log \varepsilon$ and $\log \rho_X$. Thus we may write

$$\begin{aligned} N(u_n + \varepsilon^n v) &= \varepsilon^{n+2} (\log \varepsilon)^{2n-2} (h_0) + \varepsilon^{n+2} (\log \varepsilon)^{2n-3} (h_1 + k_1) + \\ &\quad \cdots + \varepsilon^{n+2} (\log \varepsilon)^0 (h_{2n-2} + k_{2n-2}) + R_{n+1}, \\ h_j &\in \mathcal{A}^{\overline{(0,0)}_D, (-1,j)_X, \infty_B}(Z), \quad k_j \in \mathcal{A}^{0_D, \overline{(-1,j)}_X, \infty_B}(Z). \end{aligned}$$

Expanding in ρ_D , we write

$$\begin{aligned} h_j &= h'_j + h''_j, \quad h'_j \in \mathcal{A}^{0_D, (-1,j)_X, \infty_B}(Z), \quad h''_j \in \mathcal{A}^{\overline{(1,1)}_D, (-1,j)_X, \infty_B}(Z), \\ k_j &= k'_j + k''_j, \quad k'_j \in \mathcal{A}^{0_D, \overline{(-1,j)}_X, \infty_B}(Z), \quad k''_j \in \mathcal{A}^{1_D, \overline{(-1,j)}_X, \infty_B}(Z). \end{aligned}$$

By Lemma 3.14, there exist $w_j \in \mathcal{A}^{0_D, \overline{(0,j+1)}_X, \overline{(2,0)}_B}(Z)$ such that $Lw_j + \varepsilon(h'_j + k'_j) \in \mathcal{A}^{1_D, \overline{(0,j+1)}_X, \infty_B}(Z)$ for each j . Then

$$\begin{aligned} \varepsilon^{n+1} w &:= \varepsilon^{n+1} (\log \varepsilon)^{2n-2} w_0 + \cdots + \varepsilon^{n+1} (\log \varepsilon)^0 w_{2n-2} \in \mathcal{A}^{\mathcal{F}}(Z), \\ N(u_n + \varepsilon^n v + \varepsilon^{n+1} w) &= \varepsilon^{n+2} (\log \varepsilon)^{2n-2} (\tilde{f}_0 + \tilde{g}_1) + \varepsilon^{n+2} (\log \varepsilon)^{2n-3} (\tilde{f}_1 + \tilde{g}_2) + \\ &\quad \cdots + \varepsilon^{n+2} (\log \varepsilon)^0 (\tilde{f}_{2n-2} + \tilde{g}_{2n-1}) + R_{n+1} \\ \tilde{f}_j &= h''_j \in \mathcal{A}^{\overline{(1,1)}_D, (-1,j)_X, \infty_B}(Z), \\ \tilde{g}_j &= (Lw_{j-1} + \varepsilon(h'_{j-1} + k'_{j-1})) + k''_j \in \mathcal{A}^{1_D, \overline{(-1,j)}_X, \infty_B}(Z). \end{aligned}$$

Here we use that $Q(w, \cdot) \equiv 0$ by the fact that w is supported near D with vanishing \mathfrak{p}_1 component. Finally, we set $u_{n+1} = u_n + \varepsilon^n v + \varepsilon^{n+1} w$ and rewrite the leading $\log \rho_X$ terms as $\log \varepsilon - \log \rho_D$, after which we have

$$\begin{aligned} N(u_{n+1}) &= \varepsilon^{n+1} (\log \varepsilon)^{2(n+1)-3} g_0 + \varepsilon^{n+1} (\log \varepsilon)^{2(n+1)-4} (g_1 + f_1) + \\ &\quad \cdots + \varepsilon^{n+1} (\log \varepsilon)^0 (g_{2(n+1)-3} + f_{2(n+1)-3}) + \mathcal{O}(\varepsilon^{n+2}), \\ f_j &\in \mathcal{A}^{\overline{(2,j)}_D, 0_X, \infty_B}(Z), \quad g_j \in \mathcal{A}^{(2,j)_D, \overline{(0,0)}_X, \infty_B}(Z). \end{aligned}$$

Here the f_j include the leading terms from R_{n+1} , which has the form

$$R_{n+1} = \varepsilon^{n+1} \sum_{j=0}^{2n-3} (\log \varepsilon)^j Q(\tilde{u}_{1,0}, v_j) + \mathcal{O}(\varepsilon^{n+2}),$$

$$Q(\tilde{u}_{1,0}, v_j) \in \mathcal{A}^{\infty_D, 0_X, \infty_B}(Z),$$

and $\mathcal{O}(\varepsilon^{n+2})$ is used in the sense above. Note that u_{n+1} satisfies (3.35) since $Q(\tilde{u}_{m,j}, v_k) \in \mathcal{A}^{\infty_D, 0_X, \infty_B}(Z)$ and $Q(w_j, \cdot) = 0$. This completes the induction. \square

4. MODULI OF IDEAL MONOPOLES

In this section we determine the moduli space of ideal monopoles. More precisely, we consider first the effect of passing to equivalence classes with respect to the appropriate notion of gauge transformation in §4.1, and then, after discussing the moduli space of configurations $\underline{\zeta}$ in §4.2, we consider in §4.3 the effect of allowing the configuration data $\underline{\zeta}$ of an ideal monopole to vary.

4.1. Moduli of ideal monopoles for a fixed configuration of points.

The definition of an ideal monopole was given in §3.4. We shall now introduce the appropriate notions of framing and gauge transformation and define the moduli space of ideal monopoles for a fixed configuration $\underline{\zeta} = (\zeta_1, \dots, \zeta_N)$ of points. Having fixed $\underline{\zeta}$, let $Z = Z(\underline{\zeta})$ be the corresponding gluing space. Fix integers $k_0 \geq 0$, $k_j \geq 1$ ($j = 1, \dots, N$) and set

$$k_\infty = k_0 + k_1 + \dots + k_N. \quad (4.1)$$

Definition 4.1. Let $(\bar{A}, \bar{\Phi})$ be admissible monopole boundary data over S_∞ of charge k_∞ . Let ι be an ideal monopole represented by the smooth configuration (A, Φ) on Z . We denote the restriction of (A, Φ) to D and X_i by (A_D, Φ_D) and (A_i, Φ_i) , respectively. We say that ι is *framed* by $(\bar{A}, \bar{\Phi})$ if $(A, \Phi)|_{B \cap D} = (\bar{A}, \bar{\Phi})$. We say that ι is *centered* if for $j = 1, \dots, N$, the (A_j, Φ_j) represent an element of the centered moduli space $\mathcal{M}_{k_j}^c$ (but *not* $j = 0$). The group of *framed ideal gauge transformations* \mathfrak{G}_I is the group of all restrictions to $\varrho^{-1}(0)$ of elements of the group, $\mathfrak{G}_B = \{g \in C^\infty(Z(\underline{\zeta}); \text{Aut}(P)) : g|_B = 1\}$, of gauge transformations acting by the identity on B .

The moduli space of ideal monopoles (framed at S_∞) for this fixed \underline{k} and $\underline{\zeta}$ is denoted by $\mathcal{I}_{\underline{\zeta}, \underline{k}}$, and sometimes abbreviated to $\mathcal{I}_{\underline{\zeta}}$. The moduli space of centered ideal monopoles is denoted by $\mathcal{I}_{\underline{\zeta}, \underline{k}}^c$.

Proposition 4.2. *For fixed $\underline{\zeta}$, \underline{k} and framing $(\bar{A}, \bar{\Phi})$, there exist diffeomorphisms*

$$\mathcal{I}_{\underline{\zeta}} \xrightarrow{\cong} \mathcal{M}_{k_0} \times \mathcal{M}_{k_1} \times \dots \times \mathcal{M}_{k_N},$$

$$\mathcal{I}_{\underline{\zeta}}^c \xrightarrow{\cong} \mathcal{M}_{k_0} \times \mathcal{M}_{k_1}^c \times \dots \times \mathcal{M}_{k_N}^c. \quad (4.2)$$

Proof. Pick framings (admissible boundary data) $(\overline{A}_j, \overline{\Phi}_j)$ over $S_j = D \cap X_j$. For each j , composing with an ideal gauge transformation if necessary, we may assume that $(A_j, \Phi_j)|_{S_j} = (\overline{A}_j, \overline{\Phi}_j)$; fixing such data reduces the gauge group to a subgroup \mathfrak{G}_I of gauge transformations γ over $\varrho^{-1}(0)$ such that $\gamma|_{S_j}$ lies in the $U(1)$ subgroup which preserves $(\overline{A}_j, \overline{\Phi}_j)$. Thus we have an exact sequence

$$1 \longrightarrow \mathfrak{G}_{I,0} \longrightarrow \overline{\mathfrak{G}}_I \longrightarrow U(1)^{N+1} \longrightarrow 1 \quad (4.3)$$

where $\mathfrak{G}_{I,0}$ is the subgroup of gauge transformations which are the identity at all corners. Thus

$$\mathfrak{G}_{I,0} = \mathfrak{G}_0 \times \cdots \times \mathfrak{G}_N \quad (4.4)$$

and if we divide by this subgroup first we get the product

$$(C_1 \times \cdots \times C_N) \times (\mathcal{M}_{k_0} \times \mathcal{M}_{k_1} \times \cdots \times \mathcal{M}_{k_N}) \quad (4.5)$$

where the product of principal $U(1)$ -spaces $C_1 \times \cdots \times C_N$ is the moduli space (2.84) of Dirac monopoles over D , framed at all boundary faces.

The j -th $U(1)$ factor in (4.3) acts in the obvious way simultaneously on the j -th S^1 factor and the j -th moduli space (see (2.42)) in (4.5). Dividing by $U(1)^{N+1}$, we are left with the product of moduli spaces, as claimed. \square

Remark. Though diffeomorphisms (4.2) exist, they are not canonical; different choices amount to trivializations of the circles C_j in (4.5).

We next consider the global topological behavior of the parameters in our gluing construction, and the circle factors in (4.5) will play a significant role.

4.2. Configurations of points. Denote by

$$\mathcal{C}_N = \mathcal{C}_N(\mathbb{R}^3) = \{(z_1, \dots, z_N) \in (\mathbb{R}^3)^N : z_i \neq z_j, i \neq j\}$$

the configuration space of N distinct points in \mathbb{R}^3 . We denote the components of each euclidean coordinate by $z_j = (z_j^1, z_j^2, z_j^3)$.

We write

$$\mathcal{C}_N^* = \{(z_1, \dots, z_N) : z_i \neq z_j \neq 0\}$$

for the configurations of distinct *nonzero* points in \mathbb{R}^3 . By reindexing, we have an identification of this space as a subset of \mathcal{C}_{N+1} :

$$\mathcal{C}_N^* \cong \{(z_0, z_1, \dots, z_N) \in \mathcal{C}_{N+1} : z_0 = 0\}, \quad (4.6)$$

which is evidently a homotopy retraction.

This space may be decomposed by splitting off an overall scaling factor:

$$\mathcal{C}_N^* \cong \mathcal{E}_N^* \times (0, \infty)_\varepsilon, \quad (4.7)$$

where

$$\mathcal{E}_N^* = \{(\zeta_1, \dots, \zeta_N) \in \mathcal{C}_N^* : \sum_i |\zeta_i|^2 = 1\} \cong \mathcal{C}_N^* / (0, \infty)$$

represents configurations of nonzero points up to scaling, with the quotient by the scaling action

$$(0, \infty) \times \mathbb{R}^3 \ni (\varepsilon, z) \longmapsto z/\varepsilon \in \mathbb{R}^3,$$

The isomorphism (4.7) is given by

$$(z_1, \dots, z_N) = (\varepsilon^{-1}\zeta_1, \dots, \varepsilon^{-1}\zeta_N), \quad (4.8)$$

which we will frequently make use of below. There is again a homotopy retraction using (4.7), so that $\mathcal{E}_N^* \sim \mathcal{C}_N^* \sim \mathcal{C}_{N+1}$.

We partially compactify \mathcal{C}_N^* to the space

$$\overline{\mathcal{C}}_N^* := \mathcal{E}_N^* \times [0, \infty),$$

where the set $\mathcal{E}_N^* \times \{0\}$ represents configurations of points which have gone off to infinity. Note that this captures both the directions $\zeta_i / |\zeta_i| \in \mathbb{S}^2 = \partial \overline{\mathbb{R}^3}$ of the points as well as their “relative velocities”

$$(|\zeta_1|, \dots, |\zeta_N|) \in (0, 1)^N, \quad |\zeta_1|^2 + \dots + |\zeta_N|^2 = 1.$$

We are interested in scattering tangent vectors and vector fields on $\overline{\mathcal{C}}_N^*$, which are evidently generated by $\{\varepsilon^2 \partial_\varepsilon, \varepsilon \partial_{\zeta_j^i}\}$, (subject to a relation coming from the condition $\sum_j |\zeta_j|^2 = 1$). However, there is another more convenient frame. Indeed, a simple computation using (4.8) proves the following:

Proposition 4.3. *The tangent vectors*

$$\{\partial_{z_j^i} : j = 1, \dots, N, \ i = 1, 2, 3\} \subset T_{\underline{z}} \mathcal{C}_N^*$$

determine a global frame for the bundle $T\mathcal{C}_N^*$, and extend by continuity to a global frame for the scattering tangent bundle ${}^{\text{sc}}T\overline{\mathcal{C}}_N^*$, giving a trivialization

$${}^{\text{sc}}T\overline{\mathcal{C}}_N^* \cong (\mathbb{R}^3)^N \times \overline{\mathcal{C}}_N^*. \quad (4.9)$$

4.3. Moduli of ideal monopoles. We now consider the moduli space of ideal monopoles, considering their configurations as part of the moduli. To this end, we define here a provisional global version of the gluing space from §3.1 which fibers over the partial compactification $\overline{\mathcal{C}}_N^*$ of \mathcal{C}_N^* . (The gluing space will be further enlarged in §5 to fiber over a bigger parameter space involving the ideal monopoles themselves.)

Let $X = \overline{\mathbb{R}^3}$ and begin with the product

$$\mathcal{Z}_0 = X \times \overline{\mathcal{C}}_N^* = X \times \mathcal{E}_N^* \times [0, \infty).$$

From (4.7), the interior of \mathcal{Z}_0 is identified with the space $\mathbb{R}^3 \times \mathcal{C}_N^*$ with coordinates $(z, z_1, \dots, z_N) = (z, \zeta_1/\varepsilon, \dots, \zeta_N/\varepsilon)$. The *vertical diagonals* $\{z = z_j\}$, $j = 1, \dots, N$ extend to \mathcal{Z}_0 , where they meet the boundary in the corner $\partial X \times (\mathcal{E}_N^* \times \{0\})$. As before, the first step is to set

$$\mathcal{Z}_1 = [\mathcal{Z}_0; \partial X \times (\mathcal{E}_N^* \times \{0\})].$$

The front face, which we denote by \mathcal{D}_1 , is diffeomorphic to $\mathbb{S}^2 \times [0, \infty] \times \mathcal{E}_N^*$, the leftmost factors of which we identify with the space $[\overline{\mathbb{R}^3}; \{0\}]$. This blow-up resolves the vertical diagonals, in the sense that they now meet the boundary transversally over the interior of \mathcal{D}_1 . Indeed, the Euclidean

coordinate $\zeta = \varepsilon z$ extends over $\varepsilon = 0$ to a coordinate on the interior of \mathcal{D}_1 , identified with $\mathbb{R}^3 \setminus \{0\}$, and then the boundary of each vertical diagonal $\{z = z_j\}$ is the submanifold $\mathcal{P}_j = \{\zeta = \zeta_j\} \subset \mathcal{D}_1$. We blow-up the boundaries of these diagonals, setting

$$\mathcal{Z}' = [\mathcal{Z}_1; \mathcal{P}_1, \dots, \mathcal{P}_N].$$

We denote the new front faces by \mathcal{X}'_j , $j = 1, \dots, N$ and the lift of \mathcal{D}_1 by \mathcal{D}' . The lift of the original faces $X \times (\mathcal{E}_N^* \times \{0\})$ and $\partial X \times (\mathcal{E}_N^* \times [0, \infty))$ are denoted \mathcal{X}'_0 and \mathcal{B}' , respectively. We set $\mathcal{S}'_j = \mathcal{D}' \cap \mathcal{X}'_j$ for $j = 0, \dots, N$ and $\mathcal{S}'_\infty = \mathcal{D}' \cap \mathcal{B}'$.

Note that \mathcal{D}' fibers over \mathcal{E}_N^* and there is a natural identification

$$\mathcal{D}' \cong [\overline{\mathbb{R}^3} \times \mathcal{E}_N^*; \{\zeta = 0\}, \{\zeta = \zeta_1\}, \dots, \{\zeta = \zeta_N\}].$$

The fact that $\{\zeta = 0\}$ is blown up here (in fact already in \mathcal{D}_1) is consistent with the identification (4.6).

For each fixed $\underline{\zeta} = (\zeta_1, \dots, \zeta_N) \in \mathcal{E}_N^*$, the fiber of \mathcal{Z}' over $\underline{\zeta}$ is equal to the gluing space $Z = Z(\underline{\zeta})$ constructed in §3.1. We consider now the problem of obtaining families of ideal monopoles over the $\varepsilon = 0$ boundaries of \mathcal{Z}' parameterized by their configurations $\underline{\zeta}$.

Lemma 4.4. *For each $j = 0, \dots, N$, there is a canonical diffeomorphism*

$$\mathcal{X}'_j \cong \overline{\mathbb{R}^3} \times \mathcal{E}_N^*. \quad (4.10)$$

Proof. The difference function

$$\dot{\mathcal{Z}}' \cong \mathbb{R}^3 \times \mathcal{C}_N^* \ni (z, z_1, \dots, z_N) \mapsto w = z - z_j \in \mathbb{R}^3$$

extends to a smooth, bounded function on the interior of \mathcal{X}'_j . Indeed, $\zeta = \varepsilon z$ serves as an interior coordinate on \mathcal{D}_1 , after which \mathcal{X}'_j arises from the blow-up of $\zeta - \zeta_j = \varepsilon(z - z_j)$ at $\varepsilon = 0$, recovering the Euclidean coordinate $w = z - z_j$ on the interior. Along with the projection $\mathcal{X}'_j \rightarrow \mathcal{E}_N^*$, this gives a diffeomorphism $\mathring{\mathcal{X}}'_j \cong \mathbb{R}^3 \times \mathcal{E}_N^*$. As for the boundary of \mathcal{X}'_j , the function $\zeta - \zeta_j : \mathcal{D}_1 \rightarrow \mathbb{R}$ lifts to a map from \mathcal{D}' to $[\mathbb{R}^3; \{0\}]$ sending \mathcal{S}_j to the front face of the blow-up of $\{0\}$, which along with $\mathcal{D}' \rightarrow \mathcal{E}_N^*$ gives a diffeomorphism $\mathcal{S}'_j \cong \mathbb{S}^2 \times \mathcal{E}_N^*$ which is consistent with the radial compactification of $\mathbb{R}^3 \times \mathcal{E}_N^* \cong \mathring{\mathcal{X}}'_j$. \square

Lemma 4.5. *The cohomology group $H^2(\mathcal{D}'; \mathbb{Z}) = H^2(\mathring{\mathcal{D}}'; \mathbb{Z})$ splits as a direct sum*

$$H^2(\mathcal{D}'; \mathbb{Z}) \cong \mathbb{Z}^{N+1} \oplus H^2(\mathcal{E}_N^*; \mathbb{Z}), \quad H^2(\mathcal{E}_N^*; \mathbb{Z}) = \mathbb{Z}^{N(N+1)/2}$$

with the first factor representing the cohomology of the fiber $\mathbb{R}^3 \setminus \{0, \zeta_1, \dots, \zeta_N\}$ of $\mathring{\mathcal{D}}'$. In terms of the generator ω of $H^2(\mathbb{R}^3 \setminus \{0\}; \mathbb{Z}) = \mathbb{Z}$, generators of

$H^2(\mathcal{D}'; \mathbb{Z})$ are given by

$$\begin{aligned} \xi_j &= g_j^* \omega, \quad j = 0, \dots, N, \\ g_j : \mathring{\mathcal{D}}' \ni (\zeta, \zeta_1, \dots, \zeta_N) &\longmapsto \zeta - \zeta_j \in \mathbb{R}^3 \setminus \{0\} \end{aligned} \quad (4.11)$$

for the first factor, and

$$\begin{aligned} \eta_{ij} &= f_{ij}^* \omega, \quad 0 \leq i < j \leq N, \\ f_{ij} : \mathring{\mathcal{D}}' \ni (\zeta, \zeta_1, \dots, \zeta_N) &\longmapsto \zeta_i - \zeta_j \in \mathbb{R}^3 \setminus \{0\} \end{aligned} \quad (4.12)$$

for the second, where we define $\zeta_0 = 0$.

There are canonical de Rham representatives of ξ_j and η_{ij} given by the pullbacks $g_j^*(\tau)$ and $f_{ij}^*(\tau)$, where

$$\tau = *d\left(\frac{1}{r}\right) \in C^\infty(\mathbb{R}^3 \setminus \{0\}; \Lambda^2), \quad (4.13)$$

is the generator of $H^2(\mathbb{R}^3 \setminus \{0\}; \mathbb{R})$.

Remark. It is convenient to allow $i > j$ as well for the η_{ij} , which are then defined via $\eta_{ij} = -\eta_{ji}$; this follows from the action by -1 of the antipodal map in $\mathbb{R}^3 \setminus \{0\}$ on $H^2(\mathbb{R}^3 \setminus \{0\}; \mathbb{Z})$.

Proof. Consider the configuration spaces $\mathcal{C}_m = (\mathbb{R}^3)^m \setminus \Delta$ for arbitrary m , where here Δ denotes the union of all pairwise diagonals. These admit fiber bundle structures

$$\mathcal{C}_m \longrightarrow \mathcal{C}_{m-1} \quad (4.14)$$

with fiber $\mathbb{R}^3 \setminus \{z_1, \dots, z_m\}$, and it follows by induction on m and the spectral sequence for (4.14) that $H^1(\mathcal{C}_m; \mathbb{Z}) = H^3(\mathcal{C}_m; \mathbb{Z}) = 0$ for all m and

$$\begin{aligned} H^2(\mathcal{C}_{m+1}; \mathbb{Z}) &= H^2(\mathbb{R}^3 \setminus \{z_1, \dots, z_m\}; \mathbb{Z}) \oplus H^2(\mathcal{C}_m; \mathbb{Z}) \\ &= \mathbb{Z}^m \oplus \mathbb{Z}^{m(m-1)/2} = \mathbb{Z}^{m(m+1)/2}. \end{aligned}$$

Moreover, $H^2(\mathcal{C}_m; \mathbb{Z})$ is generated by the pullbacks of the generator ω of $H^2(\mathbb{R}^3 \setminus \{0\}; \mathbb{Z})$ via the maps

$$\mathcal{C}_m \longrightarrow \mathbb{R}^3 \setminus 0, \quad (z_1, \dots, z_m) \longmapsto z_i - z_j, \quad 1 \leq i < j \leq m.$$

As a manifold with boundary, \mathcal{D}' has the same cohomology as its interior, $\mathring{\mathcal{D}}'$, and the fiber bundle $\mathring{\mathcal{D}}' \longrightarrow \mathcal{E}_N^*$ is equivalent to the restriction of $\mathcal{C}_{N+2} \longrightarrow \mathcal{C}_{N+1}$ over the subspace $\mathcal{E}_N^* \subset \mathcal{C}_{N+1}$ (which is a homotopy retraction), via the relabelling $(\zeta, 0, \zeta_1, \dots, \zeta_N) = (z_1, \dots, z_{N+2})$. \square

A family of ideal monopoles represented by data on the fibers of $\mathcal{Z}' \longrightarrow \overline{\mathcal{C}}_N^*$ requires, as part of its definition, a family of $\text{SU}(2)$ -Dirac connections on \mathcal{D}' . We take a moment to consider the existence of such connections.

In the first place, we require a $\text{U}(1)$ bundle $Q' \longrightarrow \mathcal{D}'$ whose restriction to each fiber over \mathcal{E}_N^* has class $(k_0, \dots, k_N) \in H^2([\overline{\mathbb{R}^3}; \{0, \zeta_1, \dots, \zeta_N\}]; \mathbb{Z})$; that

such bundles exist follows from Lemma 4.5. We call a bundle $Q' \rightarrow \mathcal{D}'$ with

$$[Q'] = \sum_{j=0}^N k_j \xi_j \in H^2(\mathcal{D}'; \mathbb{Z}),$$

a *universal Dirac bundle*.

Next, we require a connection on Q' restricting to a Dirac connection fiberwise, which is to say that a solves the fiberwise abelian Bogomolny equation with respect to some Higgs field ϕ . Set

$$\phi_{\mathcal{D}'} = \sum_{i=0}^N k_i g_i^* \left(\frac{1}{r} \right) = \sum_{i=0}^n \frac{k_i}{|\zeta - \zeta_i|},$$

and consider the 2-form

$$F_a = \sum_{j=0}^N k_j g_j^*(\tau). \quad (4.15)$$

Restricted to a fiber, D' , of $\mathcal{D}' \rightarrow \mathcal{E}_N^*$, this satisfies $F_a|_{D'} = *d\phi_{\mathcal{D}'}|_{D'}$ by (4.13). Evidently $[F_a] = [Q'] \in H^2(\mathcal{D}'; \mathbb{R})$, so there exists a connection a with curvature F_a , and since $H^1(\mathcal{D}'; \mathbb{Z}) = 0$, the pair (Q', a) is unique up to isomorphism. In fact, pulling back a Dirac framing \bar{a} on \mathbb{S}^2 to $\mathcal{S}'_\infty \cong \mathbb{S}^2 \times \mathcal{E}_N^*$, we may assume that $a|_{\mathcal{S}'_\infty} = \bar{a}$, and two such connections are uniquely gauge equivalent. We call such an a a *universal Dirac connection*, and the association of a to a connection A on $Q' \times_{\mathrm{U}(1)} \mathrm{SU}(2)$ will be called a *universal $\mathrm{SU}(2)$ -Dirac connection*.

For each $j = 0, \dots, N$ we likewise define a (unique up to isomorphism) circle bundle with connection

$$(L_j, a_j) \rightarrow \mathcal{E}_N^*, \quad [L_j] = \sum_{i \neq j} k_i \eta_{ji} \in H^2(\mathcal{E}_N^*; \mathbb{Z}),$$

with curvature

$$F_{a_j} = \sum_{i \neq j} k_i f_{ji}^*(\tau)$$

over \mathcal{E}_N^* . We call L_j a *Gibbons-Manton circle factor*, and make the following

Definition 4.6. The (weighted) *Gibbons-Manton torus bundle* corresponding to \underline{k} is

$$\mathcal{T}_{\mathrm{GM}} = \bigoplus_{j=0}^N L_j \rightarrow \mathcal{E}_N^*. \quad (4.16)$$

We equip $\mathcal{T}_{\mathrm{GM}}$ with the connection $\bigoplus_{j=0}^N a_j$.

Proposition 4.7. Fix a degree k_j bundle $Q_{k_j} \rightarrow \mathbb{S}^2$ with any Dirac framing \bar{a}_{k_j} . Then with respect to the trivialization $\mathcal{S}'_j \cong \mathbb{S}^2 \times \mathcal{E}_N^*$, the restriction of

a universal Dirac bundle (Q', a) to \mathcal{S}'_j admits an isomorphism

$$\begin{array}{ccc} (Q', a)|_{\mathcal{S}'_j} & \xrightarrow{\cong} & \text{pr}_1^*(Q_{k_j}, \bar{a}_{k_j}) \otimes \text{pr}_2^*(L_j, a_j) \\ \downarrow & & \downarrow \\ \mathcal{S}'_j & \xrightarrow{\cong} & \mathbb{S}^2 \times \mathcal{E}_N^* \end{array} \quad (4.17)$$

Proof. This is a computation in cohomology using the generators ξ_j and η_{ij} . As noted in the proof of Lemma 4.4, the isomorphism $\mathcal{S}'_j \cong \mathbb{S}^2 \times \mathcal{E}_N^*$ is obtained by what amounts to the change of variables $(\zeta, \zeta_1, \dots, \zeta_N) \longrightarrow (\zeta' = \zeta - \zeta_j, \zeta_1, \dots, \zeta_N)$ on $\mathring{\mathcal{D}}_1 \cong (\mathbb{R}^3 \setminus \{0\}) \times \mathcal{E}_N^*$, after which \mathcal{S}'_j is identified with the front face of the blow-up of $\zeta' = 0$.

In terms of this change of coordinates, the map g_j from (4.11) becomes simply

$$g_j(\zeta', \zeta_1, \dots, \zeta_N) = \zeta' \in \mathbb{R}^3 \setminus \{0\},$$

so its extension, $\tilde{g}_j : \mathcal{D}' \longrightarrow [\overline{\mathbb{R}^3}; \{0\}]$, is identified over \mathcal{S}'_j with the projection map

$$\tilde{g}_j \cong \text{pr}_1 : \mathcal{S}'_j \cong \mathbb{S}^2 \times \mathcal{E}_N^* \longrightarrow \mathbb{S}^2.$$

On the other hand, for $i \neq j$, the map g_i can be written as

$$g_i = g_j - f_{ij} : (\zeta', \zeta_1, \dots, \zeta_N) = \zeta' - (\zeta_i - \zeta_j).$$

The extension, \tilde{g}_i , to \mathcal{D}' never vanishes on \mathcal{S}'_j (it vanishes precisely over \mathcal{S}'_i , which is disjoint), while $\zeta' \equiv 0$ over \mathcal{S}'_j as an \mathbb{R}^3 -valued function, so we obtain

$$\tilde{g}_i|_{\mathcal{S}'_j} \cong -f_{ij} \circ \text{pr}_2 : \mathcal{S}'_j \cong \mathbb{S}^2 \times \mathcal{E}_N^* \longrightarrow \mathbb{R}^3 \setminus \{0\}.$$

It follows that the cohomology class $[Q'] = \sum_i k_i \xi_i$ restricts over \mathcal{S}'_j to

$$[Q'] = k_j \xi_j - \sum_{i \neq j} k_i \eta_{ij} \cong k_j \omega \oplus [L_j] \in H^2(\mathbb{S}^2 \times \mathcal{E}_N^*) = H^2(\mathbb{S}^2) \oplus H^2(\mathcal{E}_N^*).$$

The curvature form F_a behaves similarly. Indeed, the restriction of the form (4.13) to the front face of the blow-up of the origin in \mathbb{R}^3 is the standard volume form on \mathbb{S}^2 , so it follows that

$$F_a \cong \text{pr}_1^*(F_{\bar{a}_{k_j}}) + \text{pr}_2^*(F_{a_j}),$$

and since $H^1(\mathcal{S}'_j; \mathbb{R}) = 0$, the respective bundles with connection (Q', a) and $\text{pr}_1^*(Q_{k_j}, \bar{a}_{k_j}) \otimes \text{pr}_2^*(L_j, a_j)$ are intertwined by an isomorphism which is unique up to a constant gauge transformation. \square

Proposition 4.7 shows that it is not possible to choose global framings for a Dirac monopole at the \mathcal{S}'_j , $j = 0, \dots, N$. Indeed, such a framing over \mathcal{S}'_j would amount to the existence of an isomorphism $(Q, a)|_{\mathcal{S}'_j} \cong \text{pr}_1^*(Q_{k_j}, \bar{a}_{k_j})$, which is obstructed by the nontriviality of L_j .

Proceeding instead locally, we restrict to the preimage, U' , in \mathcal{D}' of an open set $U \subset \mathcal{E}_N^*$ over which \mathcal{T}_{GM} is trivial, and then we may assume that $(a, \phi_{\mathcal{D}'})|_{U'}$ is framed at the faces $\mathcal{S}'_j \cap U'$. Any two such Dirac monopoles are identified by a gauge transformation $g \in C^\infty(U'; \text{Ad } Q')$ such that $g|_{\mathcal{S}'_\infty \cap U'} = 1$ and $g|_{\mathcal{S}'_j \cap U'}$ is constant on the fibers of $\mathcal{S}'_j \rightarrow \mathcal{E}_N^*$; such g is unique modulo the subgroup $\mathfrak{G}_0(U'; Q)$ of gauge transformations which are the identity over each \mathcal{S}'_j . Dividing by this subgroup, we are left with the action of the quotient group, identified with the fiberwise constant gauge transformations on each $\mathcal{S}'_j \rightarrow U$, or equivalently the group $C^\infty(U; \text{U}(1)^{N+1})$.

It follows that the moduli space of framed Dirac monopoles with configurations $\underline{\zeta} \in U \subset \mathcal{E}_N^*$ is a product

$$C_0 \oplus \cdots \oplus C_N \rightarrow U \quad (4.18)$$

of trivial $\text{U}(1)$ bundles C_j over U . The isomorphism (4.17) intertwines the action of the fiberwise constant gauge transformations on $\mathcal{S}'_j \rightarrow \mathcal{E}_N^*$ with the fiberwise $\text{U}(1)$ action on $L_j \rightarrow \mathcal{E}_N^*$, and thus determines a natural isomorphism

$$C_j \cong L_j|_U.$$

We conclude the following result; compare Proposition 2.15.

Proposition 4.8. *The moduli space of framed Dirac monopoles of charge \underline{k} with point configurations in \mathcal{E}_N^* is isomorphic to the Gibbons-Manton torus bundle (4.16).*

Similarly, we have

Theorem 4.9. *For fixed $\underline{k} = (k_0, \dots, k_N)$, the full moduli space, $\mathcal{I}_{\underline{k}}$, of ideal monopoles is a fiber bundle associated to \mathcal{T}_{GM} with respect to the product action of $\text{U}(1)^{N+1}$ on $\mathcal{M}_{k_0} \times \cdots \times \mathcal{M}_{k_N}$:*

$$\mathcal{I}_{\underline{k}} = \mathcal{T}_{\text{GM}} \times_{\text{U}(1)^{N+1}} (\mathcal{M}_{k_0} \times \cdots \times \mathcal{M}_{k_N}) \rightarrow \mathcal{E}_N^*. \quad (4.19)$$

Here the notation means we take the quotient of $\mathcal{T}_{\text{GM}} \times \mathcal{M}_{k_0} \times \cdots \times \mathcal{M}_{k_N}$ by the diagonal action of $\text{U}(1)^{N+1}$ acting on the right of \mathcal{T}_{GM} and on the left on the monopole moduli spaces.

Similarly,

$$\mathcal{I}_{\underline{k}}^c = \mathcal{T}_{\text{GM}} \times_{\text{U}(1)^{N+1}} (\mathcal{M}_{k_0} \times \mathcal{M}_{k_1}^c \times \cdots \times \mathcal{M}_{k_N}^c) \quad (4.20)$$

Proof. We prove the result for \mathcal{I} ; the argument for \mathcal{I}^c is similar. Fix a sufficiently small open subset $U \subset \mathcal{E}_N^*$ and a trivialization of $\mathcal{T}_{\text{GM}}|_U$. We restrict consideration to the preimage of $U \times [0, \infty)$ in \mathcal{Z}' without change of notation. Suppose (A, Φ) represents a smooth family of ideal monopoles on $\mathcal{Z}' \rightarrow U \times [0, \infty)$. In particular, $A|_{\mathcal{D}'}$ is associated to some Dirac connection a by an identification $P \cong Q \times_{\text{U}(1)} \text{SU}(2)$.

Fix charge k_j admissible monopole boundary data $(\bar{A}_j, \bar{\Phi}_j)$ on \mathbb{S}^2 for $j = 0, \dots, N$, and pull these back to \mathcal{S}'_j using the diffeomorphism $\mathcal{S}'_j \cong \mathbb{S}^2 \times U$

and left projection. By the trivialization of the $L_j|_U$, we may suppose that $(A, \Phi)|_{\mathcal{S}'_j} = (\bar{A}_j, \bar{\Phi}_j)$, composing with a gauge transformation if necessary. This reduces the gauge freedom to the subgroup $\bar{\mathfrak{G}}_I$ of gauge transformations γ over $\varrho^{-1}(U \times \{0\})$ such that $\gamma|_{\mathcal{S}'_j}$ reduces to a $U(1)$ gauge transformation which is constant on each fiber over U . This determines an exact sequence

$$1 \longrightarrow \mathfrak{G}_{I,0} \longrightarrow \bar{\mathfrak{G}}_I \longrightarrow \mathcal{H} \longrightarrow 1$$

where $\mathfrak{G}_{I,0} = \mathfrak{G}_0(\mathcal{D}') \times \mathfrak{G}_0(\mathcal{X}'_0) \times \cdots \times \mathfrak{G}_0(\mathcal{X}'_N)$ and $\mathcal{H} = C^\infty(U; U(1)^{N+1})$ may be regarded as the group of gauge transformations on a trivial $U(1)^{N+1}$ bundle over U .

Thus, we have a well-defined map

$$\begin{aligned} \mathcal{I}|_U &\longrightarrow (C_0 \oplus \cdots \oplus C_N) \times_{U(1)^{N+1}} (\mathcal{M}_{k_0} \times \cdots \times \mathcal{M}_{k_N}), \\ [(A, \Phi)] &\longmapsto [(A_{\mathcal{D}'}, \Phi_{\mathcal{D}'}), (A_0, \Phi_0), \dots, (A_N, \Phi_N)], \end{aligned}$$

with $C_0 \oplus \cdots \oplus C_N$ as in (4.18). Here we take the quotient by gauge transformations on \mathcal{Z}' restricting to the identity at \mathcal{B}' on the left, and on the right we take the quotient by $\bar{\mathfrak{G}}_I$ on the right, first by $\mathfrak{G}_{I,0}$ and then by \mathcal{H} .

Conversely, if we fix a universal framed $SU(2)$ -Dirac monopole $(A'_{\mathcal{D}'}, \Phi'_{\mathcal{D}'})$ on \mathcal{D}' over U , then

$$\begin{aligned} \mathcal{M}_{k_0} \times \cdots \times \mathcal{M}_{k_N} \times U &\longrightarrow \mathcal{I}|_U, \\ ([A_0, \Phi_0], \dots, [A_N, \Phi_N]) &\longmapsto [(A'_{\mathcal{D}'}, \Phi'_{\mathcal{D}'}), (A_0, \Phi_0), \dots, (A_N, \Phi_N)] \end{aligned}$$

gives an inverse map.

Using the fact that $C_j \cong L_j|_U$ and patching together various local trivializations for \mathcal{T}_{GM} over \mathcal{E}_N^* gives the global result. \square

Remark. We proceed to define the gluing map (1.11) on the moduli space $\mathcal{I}_{\underline{k}}$ of all ideal monopoles, as our construction works perfectly well in this setting. However, in order to obtain a local diffeomorphism onto moduli space, it will eventually be necessary to restrict to centered ideal monopoles.

In addition, the correct spaces to consider should really be the quotients $\mathcal{I}_{\underline{k}}^c / \Sigma_{\underline{k}}$, where $\Sigma_{\underline{k}}$ is the subgroup of the symmetric group on N letters which preserves the sequence (k_1, \dots, k_N) (so $\Sigma_{\underline{k}} = \Sigma_N$ if all the k_j are equal and is equal to $\{1\}$ if they are all distinct), acting by permutation on the configurations $\underline{\zeta}$ and by the obvious factor exchange on the fibers (4.2). Indeed, one expects these quotient spaces, not the $\mathcal{I}_{\underline{k}}^c$ themselves, to form the boundary hypersurfaces of the compactification of \mathcal{M}_k . However, since our gluing map is local, and we only aim to prove that it is a local diffeomorphism onto its image, it suffices to work with $\mathcal{I}_{\underline{k}}$ for simplicity.

Henceforth we will suppress the dependence on \underline{k} from the notation, writing simply \mathcal{I} or \mathcal{I}^c .

Taking the product with $[0, \infty)$, we obtain the fibration

$$\varphi : \mathcal{I} \times [0, \infty) \longrightarrow \mathcal{E}_N^* \times [0, \infty) = \bar{\mathcal{C}}_N^*. \quad (4.21)$$

With respect to the scattering structure on the base which was considered above, $\mathcal{I} \times [0, \infty)$ inherits a natural *fibred boundary structure* [MM98]. These vector fields, denoted by $\mathcal{V}_\varphi(\mathcal{I} \times [0, \infty))$, are defined as those vector fields V on $\mathcal{I} \times [0, \infty)$ for which $V\varepsilon \in \varepsilon^2 C^\infty(\mathcal{I} \times [0, \infty))$ and which are tangent to the fibers of $\varphi : \mathcal{I} \rightarrow \mathcal{E}_N^*$ over $\{\varepsilon = 0\}$. Essentially this means that V behaves as a scattering vector field along the base $\overline{\mathcal{C}}_N^*$ and an ordinary vector field along the fibers $\mathcal{M}_{k_0} \times \cdots \times \mathcal{M}_{k_N}^c$. The associated tangent bundle will be denoted ${}^\varphi T(\mathcal{I} \times [0, \infty))$, and we note that we have an exact sequence

$$0 \rightarrow T\mathcal{M}_{k_0} \times T\mathcal{M}_{k_1}^c \times \cdots \times T\mathcal{M}_{k_N}^c \rightarrow {}^\varphi T(\mathcal{I} \times [0, \infty)) \rightarrow {}^{\text{sc}} T\overline{\mathcal{C}}_N^* \cong (\mathbb{R}^3)^N \rightarrow 0, \quad (4.22)$$

where we use Proposition 4.3 to trivialize the latter space.

In fact we have a natural splitting of (4.22) since (4.21) is an associated fiber bundle to the extension of \mathcal{T}_{GM} to $\overline{\mathcal{C}}_N^*$, which comes equipped with a canonical connection. Such a connection induces a splitting of the tangent bundle sequence (4.22) for any associated bundle.

5. UNIVERSAL GLUING SPACE AND PARAMETERIZED GLUING

We now define the universal version of the gluing space from §3.1. The *universal gluing space* is

$$\mathcal{Z} = \varphi^* \mathcal{Z}' \rightarrow \mathcal{I} \times [0, \infty).$$

This pullback factors through the lift of \mathcal{Z}_0 giving a map

$$\mathcal{Z} \rightarrow \varphi^* \mathcal{Z}_0 = X \times \mathcal{I} \times [0, \infty). \quad (5.1)$$

We denote the lifts of \mathcal{D}' , \mathcal{X}'_j , \mathcal{S}'_j and \mathcal{B}' simply by \mathcal{D} , \mathcal{X}_j , \mathcal{S}_j and \mathcal{B} , respectively. As before, we fix boundary defining functions $\rho_{\mathcal{D}}$, $\rho_{\mathcal{B}}$ and $\rho_j = \rho_{\mathcal{X}_j}$ such that $\varepsilon = \rho_{\mathcal{D}} \rho_{\mathcal{X}}$, with $\rho_{\mathcal{X}} := \rho_0 \cdots \rho_N$.

Composing (5.1) with projections, we obtain the three fundamental maps

$$\mu : \mathcal{Z} \rightarrow \mathcal{I}, \quad (5.2)$$

$$\varrho : \mathcal{Z} \rightarrow \mathcal{I} \times [0, \infty), \quad (5.3)$$

$$\pi_X : \mathcal{Z} \rightarrow X. \quad (5.4)$$

The maps ϱ and π_X are b-fibrations, while μ is a smooth fiber bundle whose fibers are manifolds with corners. Indeed, each fiber of the map μ is a single parameter gluing space as defined in §3, and then ϱ and π_X restrict over each such fiber to the maps of the same names in §3.1.

To conform to the notation in §3.1, we observe a notational convention whereby fibers of μ are denoted by non-calligraphic versions of the global spaces; thus a typical fiber of \mathcal{Z} is denoted by Z , and fibers of $\mu \equiv \varrho : \mathcal{D} \rightarrow \mathcal{I}$ and $\mu \equiv \varrho : \mathcal{X}_i \rightarrow \mathcal{I}$ are denoted by D and X_i , respectively.

The relevant geometric structures on \mathcal{Z} are generated by Lie subalgebras of $\mathcal{V}(\mathcal{Z})$ as in §3.1. Within the algebra $\mathcal{V}_{\text{b}}(\mathcal{Z})$ of vector fields tangent to the boundary faces of \mathcal{Z} , we let $\mathcal{V}_\mu(\mathcal{Z})$ and $\mathcal{V}_\varrho(\mathcal{Z})$ denote the vector fields which are additionally tangent to the fibers of μ and ϱ , respectively.

Note that $\mathcal{V}_b(\mathcal{Z})$ includes vector fields in the parameter directions, while $\mathcal{V}_\mu(\mathcal{Z})$ consists solely of the b vector fields along the fibers Z , i.e., $\mathcal{V}_\mu(\mathcal{Z})|_Z \equiv \mathcal{V}_b(\mathcal{Z})$. Likewise, $\mathcal{V}_\varrho(\mathcal{Z})$ consists of the ϱ vector fields along the fibers Z .

The algebra $\mathcal{V}_\gamma(\mathcal{Z})$ is defined by $\mathcal{V}_\gamma(\mathcal{Z}) = \rho_D \rho_B \mathcal{V}_\varrho(\mathcal{Z})$ and consists of the fiberwise γ vector fields (as defined in §3.1) fiberwise. We have a filtration

$$\mathcal{V}_\gamma(\mathcal{Z}) \subset \mathcal{V}_\varrho(\mathcal{Z}) \subset \mathcal{V}_\mu(\mathcal{Z}) \subset \mathcal{V}_b(\mathcal{Z}) \subset \mathcal{V}(\mathcal{Z}).$$

The associated tangent bundles are defined as before, via $C^\infty(\mathcal{Z}; \bullet T\mathcal{Z}) = \mathcal{V}_\bullet(\mathcal{Z})$, for $\bullet \in \{b, \mu, \varrho, \gamma\}$, and the results of §3.1 carry over; namely, $\pi_X^*(\text{sc}TX)$ is naturally isomorphic to ${}^\gamma T\mathcal{Z}$, which in turn restricts over \mathcal{X}_j to the fiberwise scattering tangent bundle with respect to the fibration $X_j \rightarrow \mathcal{X}_j \rightarrow \mathcal{I}$, and rescaled restriction gives an ε -dependent isomorphism of ${}^\gamma T\mathcal{Z}|_{\mathcal{D}}$ with the fiberwise conic tangent bundle with respect to $D \rightarrow D \rightarrow \mathcal{I}$.

5.1. Bogomolny equation on \mathcal{Z} . As in §3.1 we define the γ metric

$$\tilde{g} = \pi_X^* g \quad (5.5)$$

and consider the Bogomolny operator

$$(A, \Phi) \mapsto \mathcal{B}(A, \Phi) = *F_A - d_A \Phi \in \mathcal{A}^*(\mathcal{Z}; {}^\gamma \Lambda^1 \otimes \mathfrak{p}) \quad (5.6)$$

for $P = \pi_X^* P$.

We now wish to globalize the formal construction of §3.7, to the extent possible, over $\mathcal{I} \times [0, \infty)$. The first step is the choice of a universal “pregluing configuration.”

Proposition 5.1. *Fix framed monopole data (A_j, Φ_j) with $[(A_j, \Phi_j)] \in \mathcal{M}_{k_j}$, $j = 0, \dots, N$. Then there exist neighborhoods U_j of the $[(A_j, \Phi_j)]$, closed with respect to the $U(1)$ actions on \mathcal{M}_{k_j} , and a smooth pregluing configuration (A, Φ) on $\mathcal{Z}|_{\mathcal{U} \times [0, \infty)}$ where \mathcal{U} is the set*

$$\mathcal{U} = \mathcal{T}_{\text{GM}} \times_{U(1)^{N+1}} (U_0 \times \dots \times U_N) \subset \mathcal{I}. \quad (5.7)$$

The configuration satisfies the following properties:

- (a) (A, Φ) is an approximate solution to the Bogomolny equation (5.6) with error

$$\mathcal{B}(A, \Phi) = \mathcal{O}(\rho_X \rho_D^3 \rho_B^\infty).$$

- (b) For every ideal monopole $\iota \in \mathcal{U}$, the ideal monopole represented by the restriction of (A, Φ) to $\varepsilon = 0$ in the fiber $Z = \mathcal{Z}_\iota$ is precisely ι .
(c) The γ covariant derivative d_A on ${}^\gamma \Lambda^* \otimes \mathfrak{p}$ is diagonal to infinite order at $\mathcal{D} \cup \mathcal{B}$ with respect to the splitting

$$\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_1 := \text{span}_{\mathbb{C}} \Phi \oplus \Phi^\perp$$

Remark. The need to represent monopoles by smooth families of data (A, Φ) necessitates the restriction to the neighborhoods $U_i \subset \mathcal{M}_{k_i}$; however we are able to work globally in the base directions. Note that (A, Φ) , and hence the map into moduli space \mathcal{M}_k that we later construct, is dependent on the choices of initial representatives (A_i, Φ_i) .

Remark. We refer to any configuration satisfying property (b) as a *tautological configuration*.

Proof. First, we take $\tilde{U}_i \subset \mathcal{M}_{k_i}$ to be the set of classes represented by solutions of the form $(A_i + a, \Phi_i + \phi)$ on $\overline{\mathbb{R}^3}$ for sufficiently small (a, ϕ) , where (a, ϕ) satisfy the Coulomb gauge condition $d_{A_i, \Phi_i}^*(a, \phi) = 0$ and in addition ϕ is L^2 orthogonal to $\nabla_{A_i} \Phi_i$ (see (2.56)). Thus $(A_i + a, \Phi_i + \phi)$ gives a slice for the semidirect product of the gauge group and the $U(1)$ action on \mathcal{M}_{k_i} . We then let $U_i \cong \tilde{U}_i \times U(1)$ consist of the $U(1)$ orbits of the elements in \tilde{U}_i . Defining \mathcal{U} by (5.7), it follows that we have a trivialization

$$\mathcal{U} \cong \mathcal{T}_{\text{GM}} \times (\tilde{U}_0 \times \cdots \times \tilde{U}_N). \quad (5.8)$$

Denote by $L_j \rightarrow \mathcal{U}$ the pull back of the Gibbons-Manton torus factor from \mathcal{E}_N^* to \mathcal{U} . In light of (5.8), this is equivalent to the product of pull back of L_j to \mathcal{T}_{GM} , which is canonically trivial, with $\tilde{U}_0 \times \cdots \times \tilde{U}_N$. Thus each $L_j \rightarrow \mathcal{U}$ is trivialized by our choices of (A_i, Φ_i) .

This allows us to consider global framings for ideal monopoles on $\mathcal{Z}|\mathcal{U}$. Indeed, Lemma 4.4 gives a diffeomorphism $\mathcal{X}_j \cong \overline{\mathbb{R}^3} \times \mathcal{U}$ with respect to which any universal Dirac bundle $Q \rightarrow \mathcal{D}$, obtained by pull back from \mathcal{D}' , admits an isomorphism

$$\begin{array}{ccc} Q|_{\mathcal{S}_j} & \xrightarrow{\cong} & \text{pr}_1^* Q_{k_j} \otimes \text{pr}_2^* L_j \\ \downarrow & & \downarrow \\ \mathcal{S}_j & \xrightarrow{\cong} & \mathbb{S}^2 \times \mathcal{U} \end{array} \quad (5.9)$$

By triviality of $L_j \rightarrow \mathcal{U}$, we may consider monopole framings $(\overline{A}_j, \overline{\Phi}_j)$ on $\mathcal{S}_j|\mathcal{U}$ which are pulled back from \mathbb{S}^2 .

We equip each \mathcal{X}_j with a canonical smooth family of monopoles; regarding \mathcal{X}_j as the space

$$\mathcal{X}_j \cong \overline{\mathbb{R}^3} \times \mathcal{U} \cong \overline{\mathbb{R}^3} \times \mathcal{T}_{\text{GM}} \times (\tilde{U}_0 \times \cdots \times \tilde{U}_N),$$

with the corresponding projection $\mathcal{X}_j \rightarrow \tilde{U}_j$, we endow the $\overline{\mathbb{R}^3}$ factors with the smooth family $(A_j + a, \Phi_j + \phi)$ determined by the projection to \tilde{U}_j . The framings $P|_{\mathcal{S}_j} \cong \text{pr}_1^* Q_{k_j} \times_i \text{SU}(2)$ determined by this family then identify $P|_{\mathcal{S}_j}$ with a universal Dirac bundle $Q|_{\mathcal{S}_j}$.

Next, we may pull back a universal Dirac monopole $(a, \phi_{\mathcal{D}'})$ from \mathcal{D}' to \mathcal{D} , and by associating Q to an $\text{SU}(2)$ bundle, we obtain a family of $\text{SU}(2)$ -Dirac connections $A_{\mathcal{D}}$, and an $\text{SU}(2)$ Higgs field $\Phi_{\mathcal{D}}$ satisfying $\nabla_{A_{\mathcal{D}}} \Phi_{\mathcal{D}} = 0$. Composing with a gauge transformation if necessary, we may assume that $(A_{\mathcal{D}}, \Phi_{\mathcal{D}})$ agrees with the framing of (A_i, Φ_i) (and therefore of $(A_i + a, \Phi_i + \phi)$) and the $U(1)$ orbits of these) at \mathcal{S}_i .

Finally, proceeding as in Proposition 3.9, we may produce a smooth pair (A, Φ) extending $(A_j + a, \Phi_j + \phi)$ on \mathcal{X}_j and $(A_D, 1 + \varepsilon \Phi_D)$ on \mathcal{D} , with the required properties. \square

Before proceeding with the construction of a solution from this background configuration (A, Φ) we need to discuss two topics related to analysis on \mathcal{Z} : normal operators and Sobolev spaces.

5.2. Differential and normal operators. The algebras of vector fields $\mathcal{V}_\varrho(\mathcal{Z})$ and $\mathcal{V}_\gamma(\mathcal{Z})$ give rise to algebras of differential operators: $\text{Diff}_\varrho^*(\mathcal{Z})$ and $\text{Diff}_\gamma^*(\mathcal{Z})$ are essentially the respective universal enveloping algebras of \mathcal{V}_ϱ and \mathcal{V}_γ , generated by composition with respect to the action on $C^\infty(\mathcal{Z})$. If E and F are vector bundles over \mathcal{Z} , we have similar spaces of differential operators $\text{Diff}_\varrho^*(\mathcal{Z}; E, F)$ and $\text{Diff}_\gamma^*(\mathcal{Z}; E, F)$ acting from $C^\infty(\mathcal{Z}; E)$ to $C^\infty(\mathcal{Z}; F)$.

As $\mathcal{V}_\gamma = \rho_{\mathcal{D}} \rho_{\mathcal{B}} \mathcal{V}_\varrho$, we have inclusions

$$(\rho_{\mathcal{D}} \rho_{\mathcal{B}})^k \text{Diff}_\varrho^k \subset \text{Diff}_\gamma^k,$$

though equality does not hold, the difference being in the lower order terms. For example,

$$\text{Diff}_\gamma^1 = (\rho_{\mathcal{D}} \rho_{\mathcal{B}}) \text{Diff}_\varrho^1 + \text{Diff}_\varrho^0$$

and so on.

For elements of $\mathcal{V}_\varrho(\mathcal{Z})$, restriction to \mathcal{D} makes sense as this face lies in a fiber of ϱ ; this restriction can be identified with the quotient

$$\mathcal{V}_\varrho(\mathcal{Z}) \longrightarrow \mathcal{V}_\varrho(\mathcal{Z}) / \rho_{\mathcal{D}} \mathcal{V}_\varrho(\mathcal{Z})$$

where $\rho_{\mathcal{D}} \mathcal{V}_\varrho(\mathcal{Z}) \subset \mathcal{V}_\varrho(\mathcal{Z})$ is easily seen to be an ideal. The restriction to \mathcal{X}_j is similar. Along with restriction of smooth functions, this generates maps from $\text{Diff}_\varrho^*(\mathcal{Z}; E, F)$ to differential operators on \mathcal{D} or \mathcal{X}_j , which we call the *normal operator homomorphisms* $N_{\mathcal{D}}^\varrho$ and $N_{\mathcal{X}}^\varrho$.

Proposition 5.2. *The normal operator homomorphisms define short exact sequences*

$$0 \longrightarrow \rho_{\mathcal{D}} \text{Diff}_\varrho^k(\mathcal{Z}; E, F) \longrightarrow \text{Diff}_\varrho^k(\mathcal{Z}; E, F) \xrightarrow{N_{\mathcal{D}}^\varrho} \text{Diff}_b^k(\mathcal{D}/\mathcal{I}; E, F) \longrightarrow 0$$

$$0 \longrightarrow \rho_j \text{Diff}_\varrho^k(\mathcal{Z}; E, F) \longrightarrow \text{Diff}_\varrho^k(\mathcal{Z}; E, F) \xrightarrow{N_{\mathcal{X}}^\varrho} \text{Diff}_b^k(\mathcal{X}_j/\mathcal{I}; E, F) \longrightarrow 0$$

The latter spaces denote fiberwise b differential operators with respect to the fibrations

$$\mathcal{X}_j \longrightarrow \mathcal{X}_j \longrightarrow \mathcal{I} \text{ and } \mathcal{D} \longrightarrow \mathcal{D} \longrightarrow \mathcal{I}. \quad (5.10)$$

Proof. This follows immediately from the basic claim that the quotient algebras $\mathcal{V}_\varrho(\mathcal{Z}) / \rho_{\mathcal{D}} \mathcal{V}_\varrho(\mathcal{Z})$ and $\mathcal{V}_\varrho(\mathcal{Z}) / \rho_j \mathcal{V}_\varrho(\mathcal{Z})$ may be identified with $\mathcal{V}_\varrho(\mathcal{D})$ and $\mathcal{V}_\varrho(\mathcal{X}_j)$, respectively, which are precisely the fiberwise b vector fields with respect to the fibrations (5.10). This is an easy exercise in local coordinates; for instance, near $\mathcal{D} \cap \mathcal{X}$, a general element of $\mathcal{V}_\varrho(\mathcal{Z})$ is $a(x, r, y, m)(x \partial_x -$

$r\partial_r) + \sum_j b_j(x, r, y, m)\partial_{y_j}$ with local coordinates (x, r, y) on the fiber Z and coordinate m on the base \mathcal{I} , and the quotient by $\{x(x\partial_x - r\partial_r), x\partial_y\}$ amounts to expanding the coefficients in Taylor series about $x = 0$ and throwing out terms of order $\mathcal{O}(x)$ and identifying $x\partial_x - r\partial_r$ with $-r\partial_r$, giving $-a(0, r, y, m)r\partial_r + \sum_j b_j(0, r, y, m)\partial_{y_j}$. \square

For the algebra Diff_γ^* , the normal operators at \mathcal{X}_j and \mathcal{D} are quite different from one another. On the one hand, the quotient map $\mathcal{V}_\gamma(\mathcal{Z}) \rightarrow \mathcal{V}_\gamma(\mathcal{Z})/\rho_j\mathcal{V}_\gamma(\mathcal{Z})$ is easily identified with the ordinary restriction of vector fields to \mathcal{X}_j ; in fact the quotient can be identified with the algebra $\mathcal{V}_{\text{sc}}(\mathcal{X}_j/\mathcal{I})$ of fiberwise scattering vector fields, and we have:

Proposition 5.3. *The sequence*

$$0 \rightarrow \rho_j \text{Diff}_\gamma^k(\mathcal{Z}; E, F) \rightarrow \text{Diff}_\gamma^k(\mathcal{Z}; E, F) \xrightarrow{N_\gamma^\gamma} \text{Diff}_{\text{sc}}^k(\mathcal{X}_j/\mathcal{I}; E, F) \rightarrow 0$$

is exact.

On the other hand, $[\mathcal{V}_\gamma(\mathcal{Z}), \mathcal{V}_\gamma(\mathcal{Z})] \subset \rho_{\mathcal{D}}\mathcal{V}_\gamma(\mathcal{Z})$, so in fact the quotient $\mathcal{V}_\gamma(\mathcal{Z})/\rho_{\mathcal{D}}\mathcal{V}_\gamma(\mathcal{Z})$ is an *abelian* Lie algebra, i.e., the bracket is trivial. Elements of this quotient can be regarded as families of first order, constant coefficient differential operators along the fibers of the vector bundle ${}^\gamma T\mathcal{Z} \rightarrow \mathcal{D}$, and in general the normal operator homomorphism is a map

$$N_{\mathcal{D}}^\gamma : \text{Diff}_\gamma^k(\mathcal{Z}; E, F) \rightarrow \text{Diff}_{I, \text{fib}}^k({}^\gamma T\mathcal{Z}|_{\mathcal{D}}; E, F)$$

where $\text{Diff}_{I, \text{fib}}^k({}^\gamma T\mathcal{Z}|_{\mathcal{D}})$ denotes fiberwise constant coefficient differential operators. It is then convenient to use the fiberwise Fourier transform to identify such operators with polynomials in the fibers of ${}^\gamma T^*\mathcal{Z}$, and identify composition of those differential operators with multiplication of polynomials. We denote the Fourier transform of $N_{\mathcal{D}}^\gamma$ by $\sigma_{\mathcal{D}}$.

Proposition 5.4. *The sequence*

$$0 \rightarrow \rho_{\mathcal{D}} \text{Diff}_\gamma^k(\mathcal{Z}; E, F) \rightarrow \text{Diff}_\gamma^k(\mathcal{Z}; E, F) \xrightarrow{\sigma_{\mathcal{D}}} C^\infty(\mathcal{D}; P^k({}^\gamma T\mathcal{Z}) \otimes \text{Hom}(E, F)) \rightarrow 0$$

is exact, where $P^k({}^\gamma T\mathcal{Z}) = \bigoplus_{l \leq k} S^l({}^\gamma T\mathcal{Z})$ is a sum of symmetric products of ${}^\gamma T\mathcal{Z}$, whose sections are polynomials on the fibers of ${}^\gamma T^\mathcal{Z}$.*

Proof. We again give an indication of how this works in local coordinates. The abelian lie algebra generated by $\{x^2\partial_x - xr\partial_r, x\partial_y\}$ may be identified with the one generated by the translation invariant vector fields $\{\partial_\xi, \partial_\eta\}$ on ${}^\gamma T\mathcal{Z}$, where (ξ, η) are linear coordinates on ${}^\gamma T\mathcal{Z}$ associated to the basis $\{x^2\partial_x - xr\partial_r, x\partial_y\}$. Using ordinary restriction for smooth functions, the general local vector field $a(x, r, y, m)(x^2\partial_x - xr\partial_r) + \sum_j b_j(x, r, y, m)x\partial_{y_j}$ is then identified with $a(0, r, y, m)\partial_\xi + \sum_j b_j(0, r, y, m)\partial_{\eta_j}$, and under the fiberwise Fourier transform this becomes $a(0, r, y, m)\hat{\xi} + \sum_j b_j(0, r, y, m)\hat{\eta}_j$, where $(\hat{\xi}, \hat{\eta})$ are the dual coordinates on ${}^\gamma T^*\mathcal{Z}$. \square

The content of Propositions 3.10 and 3.11 in this setting is the following. Let (A, Φ) be the pregluing configuration from Proposition 5.1, and let

$$L = D\mathcal{B}_{A,\Phi} + d_{A,\Phi}^* \quad (5.11)$$

denote the linearized Bogomolny operator at (A, Φ) , augmented by the gauge fixing operator, which we decompose relative to the splitting $\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_1$ as

$$L = \begin{pmatrix} L_0 & \mathcal{O}(\rho_{\mathcal{D}}^\infty \rho_{\mathcal{B}}^\infty) \\ \mathcal{O}(\rho_{\mathcal{D}}^\infty \rho_{\mathcal{B}}^\infty) & L_1 + \Phi_1 \end{pmatrix} \quad (5.12)$$

near $\mathcal{D} \cup \mathcal{B}$.

Proposition 5.5. *The γ normal operator at \mathcal{X}_j of L is the family of linearized gauge fixed operators*

$$N_{\mathcal{X}}^\gamma(L) = L_X = \begin{bmatrix} *d_A + \text{ad}(\Phi) & -d_A \\ -d_A^* & \text{ad}(\Phi) \end{bmatrix} \in \text{Diff}_{\text{sc}}^1(\mathcal{X}_j/\mathcal{I}; (\text{sc}\Lambda^1 \oplus \text{sc}\Lambda^0) \otimes \mathfrak{p}),$$

and the ϱ normal operator of $(\rho_{\mathcal{D}}\rho_{\mathcal{B}})^{-1}L_0$ is identified with the family of operators

$$N_{\mathcal{D}}^\varrho((\rho_{\mathcal{D}}\rho_{\mathcal{B}})^{-1}L_0) = \rho_{\mathcal{X}}\rho_{\mathcal{B}}^{-1}L_D \in \text{Diff}_{\text{b}}^1(\mathcal{D}/\mathcal{I}; {}^c\Lambda^1 \oplus {}^c\Lambda^0),$$

$$L_D = \begin{bmatrix} *Dd & -d \\ -d^* & 0 \end{bmatrix} \in \text{Diff}_{\text{c}}^1(\mathcal{D}/\mathcal{I}; {}^c\Lambda^1 \oplus {}^c\Lambda^0),$$

where we use the trivialization of \mathfrak{p}_0 over \mathcal{D} and the rescaled restriction isomorphisms $\gamma\Lambda^k\mathcal{Z} \cong {}^c\Lambda^k\mathcal{D}$, and d^* denotes the L^2 adjoint of d with respect to the family of fiberwise conic metrics $g_{\mathcal{D}}$ on the fibration $\mathcal{D} \rightarrow \mathcal{I}$.

As for the operator $L_1 + \Phi_1$, we note that L_1 is a twisting of a Dirac operator on $\gamma\Lambda^1 \oplus \gamma\Lambda^0$ by the bundle \mathfrak{p}_1 , hence its normal symbol in the sense of Proposition 5.4 is the corresponding Clifford multiplication.

Proposition 5.6. *The normal symbol of $L_1 + \Phi$ is invertible, and is given at $(x, \xi) \in \gamma T\mathcal{Z}$ by*

$$\sigma_{\mathcal{D}}(L_1 + \Phi)(x, \xi) = i\text{cl}(\xi) \otimes 1 + 1 \otimes \text{ad } \Phi_x \in \text{End}((\gamma\Lambda^1 \oplus \gamma\Lambda^0) \otimes \mathfrak{p}_1),$$

where cl is a skew adjoint Clifford action on $\gamma\Lambda^1 \oplus \gamma\Lambda^0$.

Proof. Invertibility follows from the fact that $i\text{cl}_{\text{odd}}(\xi)$ and $\text{ad } \Phi_x$ commute and are self-adjoint and skew-adjoint, respectively, with the latter nondegenerate. \square

5.3. Sobolev spaces. We first define fiberwise L^2 -based Sobolev spaces with respect to $\mu : \mathcal{Z} \rightarrow \mathcal{I}$. The bilinear form \tilde{g} in (5.5) is a fiberwise metric with respect to $\varrho : \mathcal{Z} \rightarrow \mathcal{I} \times [0, \infty)$, so to obtain a metric on the fibers of μ we set

$$\overline{g} = \tilde{g} + \pi_{[0, \infty)}^* \frac{d\varepsilon^2}{\varepsilon^2}. \quad (5.13)$$

This is a complete metric on the interior of any fiber Z of μ , and

$$L^2(Z; \overline{g}) = (\rho_{\mathcal{D}}\rho_{\mathcal{B}})^{3/2} L_{\text{b}}^2(Z),$$

where the latter space is the L^2 space defined by any b -metric on Z . For what follows we fix a fiber Z and a Hermitian vector bundle $V \rightarrow \mathcal{Z}$.

For $k, l, m \in \mathbb{N}$, let

$$H_{\mu, \varrho, \gamma}^{k, l, m}(Z; V) \ni v \iff \mathcal{V}_\gamma^{m'} \cdot \mathcal{V}_\varrho^{l'} \cdot \mathcal{V}_\mu^{k'} v \in L^2(Z; V; \bar{g}),$$

$$\forall m' \leq m, k' \leq k, l' \leq l. \quad (5.14)$$

Here the vector fields are lifted to act on sections of V by a choice of γ , ϱ , and μ connections ∇_γ , ∇_ϱ and ∇_μ , respectively, on which choices (5.14) does not depend. The subspaces (5.14) may then be equipped with inner products associated to the norms

$$\|u\|_{H_{\mu, \varrho, \gamma}^{k, l, m}}^2 = \sum_{\substack{0 \leq m' \leq m, \\ 0 \leq k' \leq k, \\ 0 \leq l' \leq l}} \left\| \nabla_\gamma^{m'} \nabla_\varrho^{k'} \nabla_\mu^{l'} u \right\|_{L^2(Z; V; \bar{g})}^2,$$

with respect to which the (5.14) are Hilbert spaces, whose topology is independent of the choice of connections. For brevity, we write $H_{\mu, \varrho}^{k, l}(Z; V) = H_{\mu, \varrho, \gamma}^{k, l, 0}(Z; V)$ and $H_{\mu, \gamma}^{k, l}(Z; V) = H_{\mu, \varrho, \gamma}^{k, 0, l}(Z; V)$. Some properties of these spaces, including multiplicativity results, are proved in Appendix A.

Remark. An alternate (and in many ways more convenient) definition of $H_\varrho^*(Z)$ and $H_\gamma^*(Z)$ in terms of pseudodifferential operators is given in Appendix D which permits the order to take any real value. However, nonnegative integer orders will suffice for our purposes.

Due to the nature of the operator (3.32), we will need to measure regularity differently near $\mathcal{D} \cup \mathcal{B}$ according to the splitting $\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_1$.

Thus, for a fixed smooth $\Phi \in C^\infty(\mathcal{Z}; \mathfrak{p})$ such that $\Phi \neq 0$ on $\mathcal{D} \cup \mathcal{B}$, the *split Sobolev spaces* are defined on a fiber Z via the norm

$$\mathcal{H}^{k, l}(Z; \mathfrak{p}) \ni v \iff \left\| \rho^{l-1} \chi v_0 \right\|_{H_{\mu, \varrho}^{k, l}} + \|\chi v_1\|_{H_{\mu, \gamma}^{k, l}} + \|(1 - \chi)v\|_{H^{k+l}} < \infty,$$

where $\rho = \rho_{\mathcal{D}} \rho_{\mathcal{B}}$, χ is a cutoff function near $\mathcal{D} \cup \mathcal{B}$ with support in the neighborhood where the splitting $\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_1 = \mathbb{C} \langle \Phi \rangle \oplus \Phi^\perp$ is defined, and $\chi v = \chi v_0 + \chi v_1$ denotes the decomposition with respect to the splitting. Thus, v_1 supports up to l derivatives of gluing type with k additional μ -derivatives in L^2 , while v_0 supports up to l derivatives of ϱ type, with up to k additional μ -derivatives in $\rho^{1-l} L^2$. In other words, near $D \cup B$,

$$\mathcal{H}^{k, l}(Z; \mathfrak{p}) \simeq \rho^{1-l} H_{\mu, \varrho}^{k, l}(Z; \mathfrak{p}_0) \oplus H_{\mu, \gamma}^{k, l}(Z; \mathfrak{p}_1).$$

The spaces $\mathcal{H}^{k, l}(Z; \mathfrak{p})$ are independent of the choice of χ , and the definition extends naturally to the spaces $\mathcal{H}^{k, l}(Z; \gamma \Lambda^* \otimes \mathfrak{p})$.

For fixed $k > 2$, these are the basic (fiberwise) Sobolev spaces with which we work, where

$$\mathcal{H}^{k, 2}(Z; \gamma \Lambda^* \otimes \mathfrak{p}) \simeq \rho^{-1} H_{\mu, \varrho}^{k, 2}(Z; \gamma \Lambda^* \otimes \mathfrak{p}_0) \oplus H_{\mu, \gamma}^{k, 2}(Z; \gamma \Lambda^* \otimes \mathfrak{p}_1)$$

supports the infinitesimal gauge transformations,

$$\mathcal{H}^{k,1}(Z; \gamma\Lambda^* \otimes \mathfrak{p}) \simeq H_{\mu, \varrho}^{k,1}(Z; \gamma\Lambda^* \otimes \mathfrak{p}_0) \oplus H_{\mu, \gamma}^{k,1}(Z; \gamma\Lambda^* \otimes \mathfrak{p}_1)$$

supports the infinitesimal monopole data, and

$$\mathcal{H}^{k,0}(Z; \gamma\Lambda^* \otimes \mathfrak{p}) \simeq \rho^1 H_{\mu}^k(Z; \gamma\Lambda^* \otimes \mathfrak{p}_0) \oplus H_{\mu}^k(Z; \gamma\Lambda^* \otimes \mathfrak{p}_1)$$

is the range of the Bogomolny map. By increasing k we increase the overall regularity, and we note that the space $\mathcal{H}^{\infty, l} = \bigcap_k \mathcal{H}^{k, l}$ includes polyhomogeneous sections with appropriate decay.

Letting the fiber Z vary, we obtain Hilbert space bundles over \mathcal{I} with fibers given by the $\mathcal{H}^{k, l}(Z; \gamma\Lambda^* \otimes \mathfrak{p})$. When working globally over any open set $\mathcal{U} \subset \mathcal{I}$, we use the Fréchet spaces

$$\mathcal{H}^{k, l}(\mathcal{Z}|_{\mathcal{U}}; \gamma\Lambda^* \otimes \mathfrak{p}) := C^{\infty}(\mathcal{U}; \mathcal{H}^{k, l}(Z; \gamma\Lambda^* \otimes \mathfrak{p})), \quad (5.15)$$

consisting of smooth sections of these Hilbert bundles.

We shall also need to restrict to a smaller range in ε ; thus we denote

$$\mathcal{H}^{k, l}(\mathcal{Z}|_{\mathcal{U} \times [0, \varepsilon_0]}; \gamma\Lambda^* \otimes \mathfrak{p}) = C^{\infty}(\mathcal{U}; \mathcal{H}^{k, l}(Z|_{\{0 \leq \varepsilon \leq \varepsilon_0\}}; \gamma\Lambda^* \otimes \mathfrak{p})),$$

which is to say the smooth sections over \mathcal{U} with values in the Sobolev space of sections on fibers Z restricted over $[0, \varepsilon_0]$ which admit extensions to (5.15). (Note that the fibers $Z \cap \{0 \leq \varepsilon \leq \varepsilon_0\}$ are not complete with respect to \bar{g} , though this will not cause any problems.)

The next result is proved in Appendix A.

Theorem 5.7. *For each $k > 2$, there is a well-defined gauge group*

$$\mathfrak{G}^k(\mathcal{Z}) = \mathcal{H}^{k, 2}(\mathcal{Z}; \text{Ad } P)$$

with Lie algebra consisting of $\mathcal{H}^{k, 2}(\mathcal{Z}; \mathfrak{p})$. This group acts on the spaces $\mathcal{H}^{k, 1}(\mathcal{Z}; \gamma\Lambda^ \otimes \mathfrak{p})$ and $\mathcal{H}^{k, 0}(\mathcal{Z}; \gamma\Lambda^* \otimes \mathfrak{p})$. Additionally, the product on $\gamma\Lambda^* \otimes \mathfrak{p}$ extends to a continuous bilinear map*

$$\mathcal{H}^{k, 1}(\mathcal{Z}; \gamma\Lambda^* \otimes \mathfrak{p}) \times \mathcal{H}^{k, 1}(\mathcal{Z}; \gamma\Lambda^* \otimes \mathfrak{p}) \longrightarrow \mathcal{H}^{k, 0}(\mathcal{Z}; \gamma\Lambda^* \otimes \mathfrak{p}) \quad (5.16)$$

5.4. Gauge fixing. Having just defined the Sobolev versions of the gauge group we will consider over \mathcal{Z} , we digress briefly to discuss the issue of gauge fixing. Fixing a sufficiently smooth configuration (A, Φ) , the infinitesimal action of the gauge group at (A, Φ) is given by the operator

$$\begin{aligned} d_{A, \Phi} : C^{\infty}(\mathcal{Z}; \mathfrak{p}) &\longrightarrow C^{\infty}(\mathcal{Z}; (\gamma\Lambda^1 \oplus \gamma\Lambda^0) \otimes \mathfrak{p}), \\ d_{A, \Phi} \eta &= (-d_A \eta, -[\Phi, \eta]). \end{aligned} \quad (5.17)$$

Note that if $(A', \Phi') = (A + a', \Phi + \phi')$, then

$$d_{A', \Phi'} = d_{A, \Phi} - (a', \phi')$$

where the latter multiplication operator acts by $(a', \phi') \cdot \eta = ([a', \eta], [\phi', \eta])$.

For smooth (A, Φ) , $d_{A, \Phi}$ admits a bounded extension

$$d_{(A, \Phi)} : \mathcal{H}^{k, 2}(\mathcal{Z}; \mathfrak{p}) \longrightarrow \mathcal{H}^{k, 1}(\mathcal{Z}; (\gamma\Lambda^1 \oplus \gamma\Lambda^0) \otimes \mathfrak{p})$$

for all k , and if $(A', \Phi') = (A + a', \Phi + \phi')$ is a perturbation with $(a', \phi') \in \mathcal{H}^{k,1}(\mathcal{Z}; (\gamma\Lambda^1 \oplus \gamma\Lambda^0) \otimes \mathfrak{p})$ then $d_{A', \Phi'}$ admits a similar extension for $k > 2$ by Theorem 5.7.

We recall that a section (a, ϕ) of $(\gamma\Lambda^1 \oplus \gamma\Lambda^0) \otimes \mathfrak{p}$ is in Coulomb gauge with respect to (A, Φ) if

$$d_{A, \Phi}^*(a, \phi) = -d_A^*a + [\Phi, \phi] = 0, \quad (5.18)$$

where the adjoints are taken with respect to the formal fiberwise L^2 pairing on sections $\gamma\Lambda^* \otimes \mathfrak{p}$ using the volume form from the γ metric \tilde{g} . In particular, $d_A^* = -*d_A*$.

This is naturally an infinitesimal condition, where (a, ϕ) are considered as elements in the tangent space to the space of configurations at (A, Φ) , and then $\text{Null}(d_{(A, \Phi)}^*)$ determines a subspace complementary to the action of the gauge group in this tangent space. However, it is also known to give local slices for the gauge action on the configuration space itself. In the present setting, this takes the form of the following result, proved in Appendix B.

Theorem 5.8. *Suppose A is a smooth (true) connection on \mathfrak{p} over \mathcal{Z} , and $\Phi \in C^\infty(\mathcal{Z}; \mathfrak{p})$, where both are diagonal to infinite order with respect to the splitting $\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_1$ near $\mathcal{D} \cup \mathcal{B}$. Fix $l > 2$ in \mathbb{N} .*

Then for any compact set $\mathcal{K} \subset \mathcal{I}$, there exists $\varepsilon_{\mathcal{K}} > 0$ such that for all sufficiently small

$$(a, \phi) \in \mathcal{H}^{l,1}(\mathcal{Z}|_{\mathcal{K} \times [0, \varepsilon_{\mathcal{K}}]}; (\gamma\Lambda^1 \oplus \gamma\Lambda^0) \otimes \mathfrak{p}),$$

there exists a unique gauge transformation

$$\gamma \in \mathfrak{G}^l(\mathcal{Z}|_{\mathcal{K} \times [0, \varepsilon_{\mathcal{K}}]})$$

such that

$$\gamma \cdot (A + a, \Phi + \phi)$$

is in Coulomb gauge with respect to (A, Φ) on \mathcal{Z} over $\mathcal{K} \times [0, \varepsilon_{\mathcal{K}}]$. Furthermore, if (a, ϕ) is additionally in $\mathcal{H}^{l+n,1}(\mathcal{Z}|_{\mathcal{K} \times [0, \varepsilon_{\mathcal{K}}]}; (\gamma\Lambda^1 \oplus \gamma\Lambda^0) \otimes \mathfrak{p})$ for any $n \geq 0$, then in fact $\gamma \in \mathfrak{G}^{l+n,2}(\mathcal{Z}|_{\mathcal{K} \times [0, \varepsilon_{\mathcal{K}}]})$ as well.

This justifies the addition of the Coulomb gauge operator to L in (5.11).

The proof of Theorem 5.8 makes use of the following invertibility result for the associated linear operator, which will also be used in §6.

Proposition 5.9. *Let (A, Φ) satisfy the hypotheses of Theorem 5.8. Then for any $k \geq 0$, the linear operator*

$$d_{A, \Phi}^* d_{A, \Phi} = \Delta_A + (\text{ad } \Phi)^*(\text{ad } \Phi) : \mathcal{H}^{k,2}(\mathcal{Z}; \mathfrak{p}) \longrightarrow \mathcal{H}^{k,0}(\mathcal{Z}; \mathfrak{p})$$

is invertible over sets of the form $\mathcal{K} \times [0, \varepsilon_{\mathcal{K}}] \subset \mathcal{I} \times [0, \infty)$, with inverse independent of k .

We also record another important fact concerning the background configuration (A, Φ) .

Proposition 5.10. *Let (A, Φ) be the pregluing configuration on $\mathcal{Z}|_{\mathcal{U}}$ of Proposition 5.1 and let $\mathcal{K} \subset \mathcal{U}$ be any compact set. Then there exists $\varepsilon_{\mathcal{K}} > 0$ such that the map*

$$\mathcal{K} \times (0, \varepsilon_{\mathcal{K}}) \ni (m, \varepsilon) \longmapsto (A, \Phi)|_{(m, \varepsilon)} \in \mathfrak{C}(\overline{\mathbb{R}^3})$$

is smooth and transverse to the orbits of the gauge group.

Proof. It suffices to show that the derivative of this map has nontrivial projections onto a complementary subspace to the infinitesimal gauge action. Recall that the infinitesimal gauge action at (A, Φ) is given by the image of the map

$$d_{A, \Phi} : \eta \longmapsto (-d_A \eta, -[\Phi, \eta])$$

and that a natural complementary subspace is the nullspace of $d_{A, \Phi}^*$, which is the Coulomb gauge slice. Projection onto $\text{Null}(d_{A, \Phi}^*)$ is given by the operator

$$\Pi = I - d_{(A, \Phi)} G d_{(A, \Phi)}^*, \quad G = (d_{(A, \Phi)}^* d_{(A, \Phi)})^{-1}.$$

By Proposition 5.9, we have a bounded operator

$$\Pi : \mathcal{H}^{\infty, 1}(\mathcal{Z}|_{\mathcal{K} \times [0, \varepsilon_{\mathcal{K}}]}; (\gamma \Lambda^1 \oplus \gamma \Lambda^0) \otimes \mathfrak{p}) \longrightarrow \mathcal{H}^{\infty, 1}(\mathcal{Z}|_{\mathcal{K} \times [0, \varepsilon_{\mathcal{K}}]}; (\gamma \Lambda^1 \oplus \gamma \Lambda^0) \otimes \mathfrak{p}),$$

representing projection onto the Coulomb gauge slice of (A, Φ) over each fiber $(m, \varepsilon) \in \mathcal{K} \times [0, \varepsilon_{\mathcal{K}}]$.

Let $V \in {}^\varphi T_{m, \varepsilon}(\mathcal{K} \times [0, \varepsilon_{\mathcal{K}}])$ be any fibered boundary vector, as defined at the end of §4. For $\varepsilon > 0$ this is any tangent vector. We may extend V as a vector field over a neighborhood of (m, ε) and apply it to (A, Φ) to get

$$(a, \phi) := V \cdot (A, \Phi) \in \mathcal{A}^*(\mathcal{Z}|_{\mathcal{K} \times [0, \varepsilon_{\mathcal{K}}]}; (\gamma \Lambda^1 \oplus \gamma \Lambda^0) \otimes \mathfrak{p}).$$

By construction of the pregluing configuration (a, ϕ) is in the Coulomb slice over $\varepsilon = 0$ so

$$\Pi(a, \phi) \in \mathcal{H}^{\infty, 1}(\mathcal{Z}|_{\mathcal{K} \times [0, \varepsilon_{\mathcal{K}}]}; (\gamma \Lambda^1 \oplus \gamma \Lambda^0) \otimes \mathfrak{p})$$

is nonvanishing at $\varepsilon = 0$, and therefore also for $\varepsilon \leq \varepsilon_V$ for some $\varepsilon_V > 0$. Minimizing ε_V over the unit sphere bundle of ${}^\varphi T(\mathcal{K} \times [0, \varepsilon_{\mathcal{K}}])$, we obtain $\varepsilon_{\mathcal{K}}$, and the result is proved. \square

5.5. True solution. Given the pregluing configuration (A, Φ) , the formal solution procedure in §3.7, applied fiber by fiber over \mathcal{U} , produces an asymptotic series for a correction $(a, \phi) \in \mathcal{A}^*(\mathcal{Z}|_{\mathcal{U} \times [0, \infty)}; (\gamma \Lambda^1 \oplus \gamma \Lambda^0) \otimes \mathfrak{p})$ at the faces \mathcal{D} and $\mathcal{X} = \cup_j \mathcal{X}_j$. To sum such a series, which has smooth coefficients in the parameters \mathcal{I} , it is necessary to restrict to a compact set. Thus let $\mathcal{K} \subset \mathcal{U}$ be any compact subset of the space (5.7).

Proposition 5.11. *Given the pregluing configuration (A, Φ) from Proposition 5.1 and a compact set $\mathcal{K} \subset \mathcal{U}$, there exists $(a, \phi) \in \mathcal{A}^{\mathcal{F}}(\mathcal{Z}|_{\mathcal{K} \times [0, \infty)}; (\gamma \Lambda^1 \oplus \gamma \Lambda^0) \otimes \mathfrak{p})$ such that $\mathcal{B}(A + a, \Phi + \phi) = \mathcal{O}(\varepsilon^\infty)$, with \mathcal{F} given as in (3.34).*

Next we remove the $\mathcal{O}(\varepsilon^\infty)$ error of our solution to the Bogomolny equation which remains after the formal construction, over a region where ε is sufficiently small. The first step is the existence of a good right parametrix for the linear operator L . The following is proved in Appendix D.6 using the pseudodifferential operator calculus developed in Appendix D. The main ingredients are the invertibility of the normal operators/symbols of L as in Propositions 5.5 and 5.6.

Proposition 5.12. *There exists a right parametrix, R to L with bounded extensions*

$$R : \mathcal{H}^{k,0}(\mathcal{Z}|_{\mathcal{U}}; (\gamma\Lambda^1 \oplus \gamma\Lambda^0) \otimes \mathfrak{p}) \longrightarrow \mathcal{H}^{k,1}(\mathcal{Z}|_{\mathcal{U}}; (\gamma\Lambda^1 \oplus \gamma\Lambda^0) \otimes \mathfrak{p}), \quad k > 2,$$

such that, for some $0 < \delta < \frac{1}{2}$,

$$LR = I - \varepsilon^\delta E,$$

where E extends to a map

$$E : \mathcal{H}^{k,0}(\mathcal{Z}|_{\mathcal{U}}; (\gamma\Lambda^1 \oplus \gamma\Lambda^0) \otimes \mathfrak{p}) \longrightarrow \mathcal{H}^{k,0}(\mathcal{Z}|_{\mathcal{U}}; (\gamma\Lambda^1 \oplus \gamma\Lambda^0) \otimes \mathfrak{p}), \quad k > 2, \quad (5.19)$$

i.e., is a smooth section over \mathcal{U} of bounded linear maps on the Hilbert bundle fibers $\mathcal{H}^{k,0}(\mathcal{Z}; (\gamma\Lambda^1 \oplus \gamma\Lambda^0) \otimes \mathfrak{p})$.

Next we set up an inverse function theorem type fixed point argument in the range space $\mathcal{H}^{k,0}(\mathcal{Z}|_{\mathcal{U}}; (\gamma\Lambda^1 \oplus \gamma\Lambda^0) \otimes \mathfrak{p})$.

Proposition 5.13. *For any compact set $\mathcal{K} \subset \mathcal{U}$, there exists $\varepsilon_{\mathcal{K}} > 0$ and $(\tilde{a}, \tilde{\phi}) \in \varepsilon^\infty \mathcal{H}^{\infty,1}(\mathcal{Z}|_{\mathcal{K} \times [0, \varepsilon_{\mathcal{K}}]}; (\gamma\Lambda^1 \oplus \gamma\Lambda^0) \otimes \mathfrak{p})$ such that $\mathcal{B}(A + a + \tilde{a}, \Phi + \phi + \tilde{\phi}) = 0$ on \mathcal{Z} over $\mathcal{K} \times [0, \varepsilon_{\mathcal{K}}]$.*

Proof. For notational convenience for the remainder of this proof, we omit the decoration $\gamma\Lambda^*$ from the bundle, understand \mathcal{Z} to be restricted over $\mathcal{K} \times [0, \infty)$, and let $u_0 = (A, \Phi)$ and $u_1 = (a, \phi)$, so that $U = u_0 + u_1$ is the the formal gauge-fixed solution of the previous section:

$$\mathcal{B}(U) \in \varepsilon^\infty \mathcal{H}^{\infty,0,0}(\mathcal{Z}; \mathfrak{p}).$$

We seek $u \in \varepsilon^\infty \mathcal{H}^{\infty,0}(\mathcal{Z}; \mathfrak{p})$ such that $\mathcal{B}(U + u) = 0$, for sufficiently small ε . Expanding the gauge-fixed Bogomolny equation into background, linear and quadratic parts, we have

$$\begin{aligned} \mathcal{B}(U + u) &= \mathcal{B}(u_0) + L_{u_0}(u_1 + u) + Q(u_1 + u) \\ &= \mathcal{B}(U) + L_{u_0}u + Q(u_1 + u) - Q(u_1). \end{aligned}$$

Here $\mathcal{B}(U) = \mathcal{O}(\varepsilon^\infty)$, Q is the zeroth order quadratic term, and $L_{u_0} = L$ is the linear operator to which we constructed a right parametrix above. It follows from the multiplicativity result in Theorem 5.7 that $u \mapsto Q(u_1 + u) - Q(u_1)$ is a bounded map from $\varepsilon^N \mathcal{H}^{k,1}(\mathcal{Z}; \mathfrak{p})$ to $\varepsilon^N \mathcal{H}^{k,0}(\mathcal{Z}; \mathfrak{p})$.

We initially seek a solution of the form $u = Rv$, $v \in \varepsilon^N \mathcal{H}^{k,0}(\mathcal{Z}; \mathfrak{p})$ for fixed N and $k > 2$, which should satisfy

$$\begin{aligned} 0 &= \mathcal{B}(U) + v - \varepsilon^\delta E v + Q(u_1 + Rv) - Q(u_1), \\ \iff v &= Tv := \varepsilon^\delta E v - \mathcal{B}(U) - Q(u_1 + Rv) + Q(u_1). \end{aligned}$$

Restricting consideration to a single fiber Z over $m \in \mathcal{K}$ for the moment, we claim that, for ε_m sufficiently small, T is a contraction mapping on a ball of sufficiently small radius in the Hilbert space $\varepsilon^N \mathcal{H}^{k,0}(Z|_{[0,\varepsilon_m]}; \mathfrak{p})$. Indeed, as a bounded operator on the latter space, $\|\varepsilon^\delta E\| \leq \varepsilon_0^\delta C$ for some $C > 0$, and $G(v) := Q(u_1 + Rv) - Q(u_1)$ vanishes at 0 along with its derivative, hence by the mean value theorem, there exists $R_m > 0$ such that

$$\|v\| \leq R_m \implies \|G(v)\| \leq \frac{1}{3} \|v\|, \quad \|G'_v v'\| \leq \frac{1}{3} \|v'\|.$$

Then taking ε_m small enough that $\varepsilon_m^\delta C < \frac{1}{3}$ and $\|\mathcal{B}(U)\| < \frac{1}{3}$, it follows that

$$T : B(0, R_m) \subset \varepsilon^N \mathcal{H}^{k,0}(Z|_{[0,\varepsilon_m]}; \mathfrak{p}) \longrightarrow B(0, R_m) \subset \varepsilon^N \mathcal{H}^{k,0}(Z|_{[0,\varepsilon_m]}; \mathfrak{p})$$

is a contraction mapping, hence has a unique fixed point v . By uniqueness of v , along with the fact that ε commutes with the linear operators E and R , it follows that in fact we may take $N \rightarrow \infty$ without altering the size of ε_m or R_m .

To see that we may also take $k \rightarrow \infty$, we proceed as follows. We have shown thus far that there exists a unique $v \in B(0, R_m) \subset \varepsilon^N \mathcal{H}^{k,0}$ such that $v - T(v) = 0$; in particular,

$$I - dT_v : \varepsilon^N \mathcal{H}^{k,0} \longrightarrow \varepsilon^N \mathcal{H}^{k,0}, \quad dT_v = \varepsilon^\delta E - Q'(u_1 + Rv) R \quad (5.20)$$

is an isomorphism. Applying the above argument in the space $\varepsilon^N \mathcal{H}^{k+1,0}$ and shrinking ε_m and R_m once if necessary, we likewise conclude that $v \in \varepsilon^N \mathcal{H}^{k+1,0}$, and that

$$I - dT_v : \varepsilon^N \mathcal{H}^{k+1,0} \longrightarrow \varepsilon^N \mathcal{H}^{k+1,0} \quad (5.21)$$

is an isomorphism which coincides with (5.20) where defined.

Now let $V \in \mathcal{V}_b(Z)$ and apply it to $v - T(v) = 0$ to obtain

$$0 = V \cdot v - V \cdot (T(v)) = (I - dT_v)(V \cdot v) + (V \cdot T)(v),$$

where

$$\begin{aligned} (V \cdot T)(v) &= [V, \varepsilon^\delta E] v - Q'(u_1 + Rv) [V, R] v \\ &\quad + (Q'(u_1) - Q'(u_1 + Rv))(V \cdot u_1) - V \cdot \mathcal{B}(U) \in \varepsilon^N \mathcal{H}^{k+1,0}. \end{aligned}$$

Indeed, $[V, \varepsilon^\delta E]$ is bounded on $\varepsilon^N \mathcal{H}^{k+1,0}$ and $[V, R]$ is bounded from $\varepsilon^N \mathcal{H}^{k+1,0}$ to $\varepsilon^N \mathcal{H}^{k+2,0}$ as shown in Appedix D, and u_1 and $\mathcal{B}(U)$ are polyhomogeneous. We conclude that $V \cdot v \in \varepsilon^N \mathcal{H}^{k+1,0}$, and since V was arbitrary, $v \in \varepsilon^N \mathcal{H}^{k+2,0}$ and the linear map

$$I - dT_v : \varepsilon^N \mathcal{H}^{k+2,0} \longrightarrow \varepsilon^N \mathcal{H}^{k+2,0}$$

is an isomorphism coinciding with (5.20) and (5.21) where defined. Proceeding inductively, we conclude that $v \in \varepsilon^N \mathcal{H}^{\infty,0}$ without any further reductions on ε_m .

Applying the above argument fiber by fiber over \mathcal{K} and taking $\varepsilon_{\mathcal{K}} = \min \{\varepsilon_m : m \in \mathcal{K}\}$ gives $v \in \varepsilon^\infty \mathcal{H}^{k,0}(\mathcal{Z}|_{\mathcal{K} \times [0, \varepsilon_{\mathcal{K}}]}; \mathfrak{p})$. \square

Combining the previous results, we have proved the following.

Theorem 5.14. *Let $\mathcal{U} \subset \mathcal{I}$ be the set (5.7), determined by choices (A_j, Φ_j) of framed monopole solutions in \mathcal{M}_{k_i} $i = 0, \dots, N$. Then for every compact set $\mathcal{K} \subset \mathcal{U}$, there exists $\varepsilon_{\mathcal{K}} > 0$ and a solution to $\mathcal{B}(A, \Phi) = 0$ on $\mathcal{Z}|_{\mathcal{K} \times [0, \varepsilon_{\mathcal{K}}]}$ which is tautological over $\varepsilon = 0$. The solution has the form $(A_0 + a, \Phi_0 + \phi)$ where (A_0, Φ_0) is smooth and $(a, \phi) \in \mathcal{A}^{\mathcal{F}}(\mathcal{Z}; \gamma\Lambda^* \otimes \mathfrak{p}) \subset \mathcal{H}^{\infty,1}(\mathcal{Z}; \gamma\Lambda^* \otimes \mathfrak{p})$, with \mathcal{F} given as in Theorem 3.15.*

In particular, (A, Φ) is smooth on $\mathcal{Z}|_{\mathcal{K} \times (0, \varepsilon_{\mathcal{K}})} = \overline{\mathbb{R}^3} \times \mathcal{K} \times (0, \varepsilon_{\mathcal{K}})$ and therefore determines a smooth map

$$\Psi : \mathcal{K} \times (0, \varepsilon_{\mathcal{K}}) \longrightarrow \mathcal{M}_k, \quad (\iota, \varepsilon) \longmapsto [(A, \Phi)|_{\iota, \varepsilon}] \quad (5.22)$$

We denote the restriction of Ψ to the sets $\mathcal{U}^c = \mathcal{U} \cap \mathcal{I}^c$ and $\mathcal{K}^c = \mathcal{K} \cap \mathcal{I}^c$ of initial data representing centered ideal monopoles by

$$\Psi^c : \mathcal{K}^c \times (0, \varepsilon_{\mathcal{K}}) \longrightarrow \mathcal{M}_k, \quad (\iota, \varepsilon) \longmapsto [(A, \Phi)|_{\iota, \varepsilon}] \quad (5.23)$$

We show below that Ψ^c is a local diffeomorphism onto its image (for possibly smaller $\varepsilon_{\mathcal{K}}$), and then compute the metric asymptotics to leading order in ε . However, it is convenient for the metric computation to allow variation in the centers of the ideal monopole data, hence our defining Ψ as we have.

Theorem 5.15. *For possibly smaller $\varepsilon_{\mathcal{K}}$, the map (5.23) is a local diffeomorphism onto its image.*

Proof. Since $\mathcal{K} \times (0, \varepsilon_{\mathcal{K}})$ and \mathcal{M}_k are both smooth manifolds of dimension $4k$, it suffices to verify that Ψ^c is an immersion. Letting $V \in T_p(\mathcal{K} \times (0, \varepsilon_{\mathcal{K}}))$ be a nonzero tangent vector, we may regard it as an element of ${}^{\varphi}T_p(\mathcal{K} \times [0, \varepsilon_{\mathcal{K}}])$ and extend it locally to a vector field \tilde{V} . The derivative is given by

$$D\Psi_p^c V = [(\iota_{\tilde{V}} F_A, \nabla_{\tilde{V}} \Phi)|_p] \in T_{[A, \Phi]} \mathcal{M}_k,$$

where we have used the connection A to lift \tilde{V} to act on (A, Φ) . As (A, Φ) are a smooth family of solutions to $\mathcal{B}(A, \Phi) = 0$, it follows that $(a, \phi) := \tilde{V} \cdot (A, \Phi)$ satisfies $D\mathcal{B}_{A, \Phi}(a, \phi) = 0$. To see that $[(a, \phi)] \neq 0 \in T_{[A, \Phi]} \mathcal{M}_k$, it suffices to verify that (a, ϕ) are transverse to the gauge orbit. However, shrinking $\varepsilon_{\mathcal{K}}$ if necessary, this follows from Proposition 5.10 since (A, Φ) satisfies the tautological property. Thus, for some $\varepsilon_{\mathcal{K}} > 0$,

$$V \neq 0 \implies d\Phi_p V \neq 0, \quad p \in \mathcal{K} \times (0, \varepsilon_{\mathcal{K}}),$$

so that Ψ^c is an immersion. \square

6. THE METRIC

In this section we show how to compute the monopole metric to leading order in ε at a monopole $m(\varepsilon)$, say, which is in the image of our ‘gluing map’ Ψ from Theorem 5.14. In particular we shall prove Theorem 1.3.

Let Z be the gluing space of §3 used in the construction of a 1-parameter family of monopoles with the configuration data $\underline{\zeta} = (\zeta_1, \dots, \zeta_N)$ fixed. The formal solution constructed in that section (and then improved to a genuine solution in §5.5) is of the general form

$$(A, \Phi) = (\overline{A}, \overline{\Phi}) + (a, \phi) \quad (6.1)$$

where $(\overline{A}, \overline{\Phi})$ is a smooth pre-gluing configuration⁴

$$(a, \phi) \in \mathcal{A}_s(Z), \quad (6.2)$$

this space $\mathcal{A}_s(Z)$ being the conormal space given in Theorem 3.15 (cf. (3.34)):

$$\begin{aligned} F_B &= \{(2, 0), (3, 1), \dots\}, \\ F_D &= \{(2, 0), (3, 1), \dots\}, \\ F_X &= \{(1, 0), (2, 0), (3, 1), \dots\}. \end{aligned} \quad (6.3)$$

(The index sets F_B and F_D start at order 2 because of the definition of pregluing configuration.) In this section we generally suppress the coefficient bundles $\gamma\Lambda = \gamma\Lambda^0 \oplus \gamma\Lambda^1$ as well as \mathfrak{p} . We recall that the component in \mathfrak{p}_1 of (a, ϕ) is rapidly decreasing at $B \cup D$.

It will also be convenient to denote the restriction of (A, Φ) to the fibre $\varrho^{-1}(\varepsilon)$ by $(A(\varepsilon), \Phi(\varepsilon))$. We apply this convention similarly to other data, so for example we shall write $\mathcal{M}_k(\varepsilon)$ for the framed moduli space of monopoles of charge k on $\varrho^{-1}(\varepsilon)$ and shall denote by $m(\varepsilon)$ the point of $\mathcal{M}_k(\varepsilon)$ represented by $(A(\varepsilon), \Phi(\varepsilon))$. Recall that for positive ε all fibres $\varrho^{-1}(\varepsilon)$ are canonically identified with the original $\overline{\mathbb{R}}^3$, so we can equally regard $m(\varepsilon)$ as a 1-parameter family in \mathcal{M}_k .

For the purposes of this section, it is convenient to allow uncentred ideal monopoles as initial data in our construction, which is to say we consider the map (5.22); instead, we restrict the configuration data to $\underline{\zeta}$. For clarity, we denote this restricted gluing map by

$$\psi = \Psi|_{\underline{\zeta}} : U_0 \times U_1 \times \dots \times U_N \times (0, \varepsilon_0) \longrightarrow \mathcal{M}_k \quad (6.4)$$

for bounded open neighbourhoods U_j of m_j in \mathcal{M}_{k_j} .

Now consider a smooth one-parameter family of ideal monopoles ι_t . As in Prop. 5.1 we may also assume that $\iota_t|_D$ is independent of t . We assume $\iota_0 = \iota$ is a centred ideal monopole and that $\iota_t \in U_0 \times U_1 \times \dots \times U_N$ is in the domain of (6.4). Let

$$u_j = \frac{d}{dt}(\iota_t|_{X_j}) \Big|_{t=0}. \quad (6.5)$$

⁴Elsewhere $(\overline{A}, \overline{\Phi})$ has typically been used for framings. We hope the reader will forgive us for the present change of usage.

This is the tangent vector to $(A_j, \Phi_j) = \iota|X_j$ in the 1-parameter family ι_t . We may assume that u_j is in Coulomb Gauge with respect to (A_j, Φ_j) so that

$$L_j u_j = 0 \text{ on } \mathring{X}_j \quad (6.6)$$

where as before, L_j is the linearization/gauge-fixing operator associated to (A_j, Φ_j) (cf. §2.6).

Our 1-parameter family ι_t gives rise to a smooth 1-parameter family $(\bar{A}_t, \bar{\Phi}_t)$ of pregluing configurations and hence a smooth 1-parameter family of solutions

$$(A_t, \Phi_t) = (\bar{A}_t, \bar{\Phi}_t) + (a_t, \phi_t). \quad (6.7)$$

Then the derivative $D\psi$ of ψ assigns to (u_0, \dots, u_n) the field

$$u(\varepsilon) = \left(\frac{d}{dt}(A_t, \Phi_t) \Big|_{t=0} \right) \Big|_{\varrho^{-1}(\varepsilon)}. \quad (6.8)$$

on $\varrho^{-1}(\varepsilon)$ which represents a tangent vector $[u(\varepsilon)]$ to $\mathcal{M}_k(\varepsilon)$.

Lemma 6.1. *If u is as defined in (6.8), then*

$$u \in \mathcal{A}'_s(Z), \quad u|X_j = u_j, \quad (6.9)$$

where the index sets of $\mathcal{A}'_s(Z)$ are

$$\begin{aligned} F'_B &= \{(2, 0), (3, 1), \dots\}, \\ F'_D &= \{(2, 0), (3, 1), \dots\}, \\ F'_X &= \{(0, 0), (1, 0), (2, 0), (3, 1), \dots\}. \end{aligned} \quad (6.10)$$

Proof. Because the framings at the corners are independent of t , we can take the restriction to D of the pregluing configuration also to be independent of t , as in the proof of Prop. 5.1. Then we have

$$\frac{d}{dt}(A_t, \Phi_t) \Big|_{t=0} \in \rho_B^2 \rho_D^2 C^\infty(Z) \subset \mathcal{A}'_s(Z) \quad (6.11)$$

and its restriction to X_j is just the variation $u_j = (\dot{A}_j, \dot{\Phi}_j)$ of $\iota_0|X_j$. The variation in (a, ϕ) lies in $\mathcal{A}_s(Z) \subset \mathcal{A}'_s(Z)$ and so the result follows. \square

To compute the length-squared of $[u(\varepsilon)]$ with respect to the monopole metric $G(\varepsilon)$ on $\mathcal{M}_k(\varepsilon)$, we need to replace u by a representative of the same element of $T_{m(\varepsilon)}\mathcal{M}_k(\varepsilon)$ but which is in Coulomb gauge with respect to $(A(\varepsilon), \Phi(\varepsilon))$. This is accomplished in the following:

Lemma 6.2. *Given the above data, there exists an infinitesimal gauge transformation ξ over Z such that*

$$d_{A, \Phi}^*(u - d_{A, \Phi}\xi) = 0, \quad (6.12)$$

where ξ is smooth over the fibers $\varrho^{-1}(\varepsilon)$ for $\varepsilon > 0$ and vanishing over $\varrho^{-1}(0)$.

Proof. (Cf. also Proposition 5.10.) It is convenient in this discussion to use the notation $f = O(\rho_B^a \rho_D^b \rho_X^c)$ to mean that f has a polyhomogeneous conormal expansion on Z with smooth index sets of the kinds that have appeared throughout this paper, with the lower bounds $(a, 0)_B$, $(b, 0)_D$ and $(c, 0)_X$. Thus $u = O(\rho_B^2 \rho_D^2)$ by virtue (6.9). Recall also the notation

$$d_{A,\Phi}\xi = (d_A\xi, [\Phi, \xi])$$

for the infinitesimal action of the gauge group on monopole configurations.

By construction, on each fibre u is in Coulomb gauge with respect to $(\bar{A}, \bar{\Phi})$,

$$d_{\bar{A}, \bar{\Phi}}^* u|_{\varrho^{-1}(\varepsilon)} = 0 \quad (6.13)$$

and

$$(d_{A,\Phi}^* u - d_{\bar{A}, \bar{\Phi}}^* u)|_{\varrho^{-1}(\varepsilon)} = \{(a, \phi), u\}|_{\varrho^{-1}(0)} \quad (6.14)$$

by (6.1), where $\{\cdot, \cdot\}$ on the RHS is some bilinear operation with smooth coefficients. Hence

$$d_{A,\Phi}^* u = O(\varepsilon \rho_B^4 \rho_D^3). \quad (6.15)$$

The equation

$$d_{A,\Phi}^*(u - d_{A,\Phi}\xi) = 0 \quad (6.16)$$

can be solved with $\xi = O(\varepsilon \rho_B^2 \rho_D)$ and $d_{A,\Phi}\xi$ therefore $O(\varepsilon \rho_B^3 \rho_D^2)$. Indeed, we may first construct a formal solution, proceeding as in §3.7, giving the asymptotic estimate, and then remove the rapidly vanishing error using Proposition 5.9. The above estimates give that ξ vanishes on $\varrho^{-1}(0)$, completing the proof. \square

With ξ from the Lemma, define

$$\tilde{u} = u - d_{A,\Phi}\xi \quad (6.17)$$

Then for each positive ε ,

$$[\tilde{u}(\varepsilon)] = [u(\varepsilon)] \in T_{m(\varepsilon)}\mathcal{M}_k(\varepsilon) \quad (6.18)$$

In this way we define a family of mappings

$$\tilde{f}_\varepsilon : T_{m_0}\mathcal{M}_{k_0} \times T_{m_1}\mathcal{M}_{k_1} \times T_{m_N}\mathcal{M}_{k_N} \longrightarrow T_{m(\varepsilon)}\mathcal{M}_k(\varepsilon) \quad (6.19)$$

where

$$\tilde{f}_\varepsilon(u_0, u_1, \dots, u_N) = \tilde{u}(\varepsilon) \quad (6.20)$$

which represents the differential, $D\psi$, of (6.4), the advantage being that

$$\tilde{u}(\varepsilon) \neq 0 \Leftrightarrow [\tilde{u}(\varepsilon)] \neq 0. \quad (6.21)$$

We note

Lemma 6.3. *The map \tilde{f}_ε is an isomorphism for sufficiently small $\varepsilon > 0$. In particular Ψ is a local diffeomorphism for sufficiently small $\varepsilon > 0$.*

Proof. If u_1 , say, is non-zero, then from the estimates in Lemma 6.2, it follows that $\tilde{u}(\varepsilon) \neq 0$ near X_1 . By (6.21), this means that $[\tilde{u}(\varepsilon)] \neq 0$ and so the map is injective. Since \tilde{f}_ε is a linear map between vector spaces of the same dimension, it is an isomorphism. \square

We can now prove Theorem 1.3. We have seen in Prop. 2.11 that for a centred monopole m , the metric on $T_m\mathcal{M}_k$ decomposes canonically as an orthogonal direct sum of \mathbb{R}^3 , with $2\pi k$ times the euclidean metric, and $T_m\mathcal{M}_k^c$. It is therefore enough to prove that the metric G_ε on $T_{m(\varepsilon)}\mathcal{M}_k(\varepsilon)$ is approximately equal to the product metric on the product of $T_{m_j}\mathcal{M}_{k_j}$.

Theorem 6.4. *Let f_ε be as above. Then f_ε is an approximate isometry.*

Proof. By the previous lemmas,

$$\tilde{u} = O(\rho_B^2 \rho_D^2) \quad (6.22)$$

and

$$\tilde{u}|_{X_j} = u_j. \quad (6.23)$$

Hence the pointwise length-squared $|\tilde{u}(\varepsilon)|^2$ on $\varrho^{-1}(0)$ is $O(\rho_B^4 \rho_D^4)$ and its restriction to each X_j is $|u_j|^2$.

In order to do the integration, we have to multiply by the lift of the euclidean density $d\mu_e$ to Z . Since $\gamma^*T = \rho_B \rho_D \varrho^*T$, the lift of the euclidean density has the form $\rho_B^{-3} \rho_D^{-3} \mu_b$ where μ_b is a smooth positive section of $\Lambda^3 \varrho^*T^*$, whose restriction to each X_j is the euclidean volume element $d\mu_j$. Hence the density

$$|\tilde{u}|^2 d\mu_e = O(\rho_B \rho_D) \quad (6.24)$$

and its restriction to each of the X_j is the density $|u_j|^2 d\mu_j$. Performing the integration, we see that

$$G_\varepsilon(\tilde{u}(\varepsilon), \tilde{u}(\varepsilon)) = \sum_{j=0}^N G_j(u_j, u_j) + O(\varepsilon) \quad (6.25)$$

as required. \square

6.1. Infinitesimal translations. We now consider the differential of the gluing map Ψ or Ψ^c with respect to variation in the base parameters, $\bar{\mathcal{C}}_N^*$. In light of Proposition 4.3, it suffices to consider translations.

Thus, fix $j \in \{1, \dots, N\}$, let $V = \xi \cdot \partial_{z_j} = \xi_1 \partial_{z_j^1} + \xi_2 \partial_{z_j^2} + \xi_3 \partial_{z_j^3} \in \mathcal{V}_{\text{sc}}(\bar{\mathcal{C}}_N^*)$ be an infinitesimal translation along the j th Euclidean factor. In particular, as an ordinary vector field, V vanishes at $\mathcal{E}_N^* = \partial \bar{\mathcal{C}}_N^*$.

In order to lift this to \mathcal{Z} , we first lift V to \mathcal{Z}' using the product structure on the interior, $\mathcal{Z}' = \mathbb{R}^3 \times \mathcal{C}_N^*$, and extension by continuity. As discussed at the end of §4.3, we may then use the connection on $\mathcal{I} \times [0, \infty)$ induced by the canonical connection on the Gibbons-Manton torus bundle to obtain a horizontal lift $\tilde{V} \in \mathcal{V}(\mathcal{Z})$.

Lemma 6.5. \tilde{V} vanishes at \mathcal{D} and \mathcal{X}_i , $i \neq j$, and with respect to the identification of \mathcal{X}_j with $\overline{\mathbb{R}^3} \times \mathcal{I}$ induced by Lemma 4.4, we have

$$\tilde{V}|_{\mathcal{X}_j} = (-\xi \cdot \partial_z, 0), \quad \mathcal{X}_j \cong \overline{\mathbb{R}^3} \times \mathcal{I}. \quad (6.26)$$

Proof. As discussed in the proof of Lemma 4.4, coordinates on \mathcal{Z}' near the interior of \mathcal{X}'_i are given by $(\varepsilon, w_i, \zeta_1, \dots, \zeta_N)$, where $w_i = z - z_i$. The lift, \tilde{V}' , of V to \mathcal{X}'_i is therefore given by

$$\tilde{V}' = \varepsilon \xi \cdot \partial_{\zeta_j} - \delta_{ij} \xi \cdot \partial_{w_i},$$

which vanishes at $\varepsilon = 0$ if $i \neq j$ and gives a vector field like (6.26) otherwise, with respect to $\mathcal{X}'_j \cong \overline{\mathbb{R}^3} \times \mathcal{E}_N^*$. It is similarly easy to verify that $\tilde{V}' = \mathcal{O}(\varepsilon)$ near \mathcal{D}' and \mathcal{B}' .

Denoting the further lift of \tilde{V}' to \mathcal{Z} by \tilde{V} using the Gibbons-Manton connection, we note that any component of \tilde{V} in the parameter directions vanishes over $\varepsilon = 0$, since this is the lift with respect to a smooth connection of V , which vanishes there; and (6.26) follows at once. \square

Now let (A, Φ) represent a solution to $\mathcal{B}(A, \Phi) = 0$ on $\mathcal{Z}|_{\mathcal{K} \times [0, \varepsilon_{\mathcal{K}}]}$ as obtained in the previous section. To compute the variation in (A, Φ) with respect to \tilde{V} , we may use the connection A itself to differentiate (recall that, while the subsequent modifications of the pregluing connection were in the fiber directions, i.e., sections of ${}^{\gamma}\Lambda^1 \otimes \mathfrak{p}$, A is nevertheless a full connection on $P \rightarrow \mathcal{Z}$), which yields

$$\tilde{V} \cdot (A, \Phi) = (\iota_{\tilde{V}} F_A, \nabla_{\tilde{V}} \Phi).$$

In light of Lemma 6.5, we obtain the following result:

Corollary 6.6. *The variation $\tilde{V} \cdot (A, \Phi)$ vanishes at \mathcal{D} and \mathcal{X}_i for $i \neq j$, while*

$$\tilde{V} \cdot (A, \Phi)|_{\mathcal{X}_j} = (-\iota_{\xi} F_A, -\nabla_{\xi} \Phi) = -\tau_{\xi},$$

where τ_{ξ} was introduced in §2.8.

In particular, this is equivalent modulo $\mathcal{O}(\varepsilon)$ to a variation of the ideal monopole family by the infinitesimal translation $-\xi$ in the j th factor, the metric evaluation of which was considered in the previous section.

APPENDIX A. SOBOLEV SPACES

In this section we prove the fundamental multiplicativity results for the Sobolev spaces introduced in §5.3. It will be sufficient to work fiberwise over \mathcal{I} , so for the remainder of the section we consider a fixed fiber, Z , of $\mu : \mathcal{Z} \rightarrow \mathcal{I}$.

Lemma A.1. *Let $\rho := \rho_D \rho_B$. For $k > 2$, $l \geq 0$, multiplication of smooth functions extends to bilinear maps on the Sobolev spaces of Definition 5.14:*

$$\rho^\alpha H_{\mu,\varrho}^{k,l}(Z;\bar{g}) \times \rho^\alpha H_{\mu,\varrho}^{k,l}(Z;\bar{g}) \longrightarrow \rho^{2\alpha+3/2} H_{\mu,\varrho}^{k,l}(Z;\bar{g}), \quad (\text{A.1a})$$

$$\rho^\alpha H_{\mu,\varrho}^{k,l}(Z;\bar{g}) \times \rho^\beta H_{\mu,\gamma}^{k,l}(Z;\bar{g}) \longrightarrow \rho^{\alpha+\beta+3/2} H_{\mu,\gamma}^{k,l}(Z;\bar{g}), \quad (\text{A.1b})$$

$$\rho^\beta H_{\mu,\gamma}^{k,l}(Z;\bar{g}) \times \rho^\beta H_{\mu,\gamma}^{k,l}(Z;\bar{g}) \longrightarrow \rho^{2\beta-l+3/2} H_{\mu,\varrho}^{k,l}(Z;\bar{g}). \quad (\text{A.1c})$$

Proof. Let $g_b = \rho^{-2}\bar{g}$ be the associated b -metric on Z . Then (\mathring{Z}, g_b) is a complete Riemannian 4-manifold, which enjoys the same Sobolev embedding results as \mathbb{R}^4 with respect to derivatives which are bounded with respect to g_b , i.e., with respect to b -derivatives $\mathcal{V}_b(Z) \equiv \mathcal{V}_\varrho(Z)$. In particular, $H_\mu^k(Z; g_b)$ is an algebra for $k > 2$, and distributing derivatives via the Liebnitz formula, it follows that, for $l \geq 0$, the spaces $H_{\mu,\varrho}^{k,l}(Z; g_b)$ and $H_{\mu,\gamma}^{k,l}(Z; g_b)$ are algebras, and

$$\begin{aligned} \rho^{\alpha'} H_{\mu,\varrho}^{k,l}(Z; g_b) \times \rho^{\beta'} H_{\mu,\varrho}^{k,l}(Z; g_b) &\longrightarrow \rho^{\alpha'+\beta'} H_{\mu,\varrho}^{k,l}(Z; g_b), \\ \rho^{\alpha'} H_{\mu,\gamma}^{k,l}(Z; g_b) \times \rho^{\beta'} H_{\mu,\gamma}^{k,l}(Z; g_b) &\longrightarrow \rho^{\alpha'+\beta'} H_{\mu,\gamma}^{k,l}(Z; g_b). \end{aligned}$$

The results above then follow from the identity $\rho^{\alpha'} L^2(Z; g_b) = \rho^\alpha L^2(Z; \bar{g})$ where $\alpha' = \alpha + \frac{3}{2}$, and the inclusions

$$\rho^\alpha H_{\mu,\varrho}^{k,l}(Z; \bar{g}) \subset \rho^\alpha H_{\mu,\gamma}^{k,l}(Z; \bar{g}), \quad \rho^\beta H_{\mu,\gamma}^{k,l}(Z; \bar{g}) \subset \rho^{\beta-l} H_{\mu,\varrho}^{k,l}(Z; \bar{g}),$$

which in turn follow from the fact that $\mathcal{V}_\gamma(Z) \ni X = \rho \tilde{X}$ for $\tilde{X} \in \mathcal{V}_\varrho(Z)$. \square

Proof of Theorem 5.7. With respect to the splitting $\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_1$, the product on $\Lambda^* \otimes \mathfrak{p}$ decomposes as

$$[u, v]_0 = [u_1, v_1], \quad [u, v]_1 = [u_0, v_1] + [u_1, v_0].$$

Boundedness of the products $\mathcal{H}^{k,l',\beta} \times \mathcal{H}^{k,l,\beta} \longrightarrow \mathcal{H}^{k,l,\beta}$ for $l' \geq l$ then follows from (A.1b) and (A.1c)

For the gauge group, we work in the universal enveloping algebra $\mathcal{U}(\mathfrak{p})$, here identifiable with 2×2 complex matrices locally. Near $D \cup B$, we have a splitting

$$\mathcal{U}(\mathfrak{p}) = \mathcal{U}(\mathfrak{p})_0 \oplus \mathcal{U}(\mathfrak{p})_1$$

consistent with the splitting of $\mathfrak{p} \subset \mathcal{U}(\mathfrak{p})$; indeed, we may take $\mathcal{U}(\mathfrak{p})_0$ and $\mathcal{U}(\mathfrak{p})_1$ to be the diagonal and anti-diagonal matrices, respectively. The product in $\mathcal{U}(\mathfrak{p})$ then decomposes as

$$(uv)_0 = u_0 v_0 + u_1 v_1, \quad (uv)_1 = u_0 v_1 + u_1 v_0,$$

and it follows from Lemma A.1, (A.1a)–(A.1c) that $\mathcal{H}^{k,2,\beta}(Z; \mathcal{U}(\mathfrak{p}))$ is an algebra. Adjoining a unit and exponentiating in the algebra $1 + \mathcal{H}^{k,2,\beta}(Z; \mathcal{U}(\mathfrak{p}))$, we obtain the gauge group $\mathcal{H}^{k,2,\beta}(Z; \text{Ad } P)$, with Lie algebra $\mathcal{H}^{k,2,\beta}(Z; \mathfrak{p})$ as claimed. The action of this group on the spaces $\mathcal{H}^{k,l,\beta}(Z; \Lambda^* \otimes \mathfrak{p})$ follows from boundedness of the infinitesimal action. \square

A.1. Sobolev spaces for X_i . In appendix C we require hybrid b/scattering Sobolev spaces on the fibers X_i . Thus let

$$H_{b,sc}^{k,l}(X_i; V) \ni v \iff \mathcal{V}_b^{k'} \cdot \mathcal{V}_{sc}^{l'} v \in L^2(X_i; V; g), \quad \forall k' \leq k, l' \leq l.$$

Here $L^2(X_i; V; g)$ is defined with respect to the induced metric on X_i ; from Proposition 3.3 this is the Euclidean metric with respect to the identification $X_i \cong \overline{\mathbb{R}^3}$. The spaces $H_{b,sc}^{k,l}(X_i; V)$ are Hilbert spaces with respect to inner products constructed from any choices of b and scattering connections on V . We consider also weighted versions $\rho^\alpha H_{b,sc}^{k,l}(X_i; V)$, where $\rho = \rho_D$. The next result is proved in [Kot15a].

Proposition A.2. *If $\alpha' \geq \alpha$, $k' \geq k$ and $l' \geq l$, then*

$$\rho^{\alpha'} H_{b,sc}^{k',l'}(X_i; V) \subset \rho^\alpha H_{b,sc}^{k,l}(X_i; V).$$

Furthermore, if $\alpha' > \alpha$ and either $k' > k$ or $l' > l$, then the inclusion is compact. If $\alpha \geq \beta + l$, then

$$\rho^\alpha H_{b,sc}^{k,l}(X_i; V) \subset \rho^\beta H_b^{k+l}(X_i; V).$$

For $V = \mathfrak{p}$, with the associated splitting $\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_1$ near ∂X_i , we define the *split Sobolev spaces* (cf. §5.3)

$$H_{b/sc}^{k,l}(X_i; \mathfrak{p}) \simeq H_b^{k+l}(X_i; \mathfrak{p}_0) \oplus H_{b,sc}^{k,l}(X_i; \mathfrak{p}_1)$$

via the norm

$$H_{b/sc}^{k,l}(X_i; \mathfrak{p}) \ni v \iff \|\chi v_0\|_{H_b^{k+l}} + \|\chi v_1\|_{H_{b,sc}^{k,l}} + \|(1-\chi)v\|_{H^{k+l}} < \infty,$$

where χ is a smooth cutoff supported near $\partial X_i = D$. We denote weighted versions of these spaces by

$$\rho^{\alpha,\beta} H_{b/sc}^{k,l}(X_i; \mathfrak{p}) \simeq \rho^\alpha H_b^{k+l}(X_i; \mathfrak{p}_0) \oplus \rho^\beta H_{b,sc}^{k,l}(X_i; \mathfrak{p}_1),$$

where $\rho = \rho_D$.

APPENDIX B. COULOMB GAUGE

This section is devoted to a proof of Theorem 5.8. Combining (5.17) and (5.18), the condition that $\gamma \cdot (A + a, \Phi + \phi)$ be in Coulomb gauge with respect to (A, Φ) amounts to the condition $G(a, \phi, \gamma) = 0$, where

$$G(a, \phi, \gamma) = d_A^*(a - (d_{A+a}\gamma)\gamma^{-1}) - \text{ad } \Phi(\gamma(\Phi + \phi)\gamma^{-1}). \quad (\text{B.1})$$

Lemma B.1. *For $k \geq 3$, (B.1) extends to a differentiable map*

$$G : \mathcal{H}^{k,1}(\mathcal{Z}; \Lambda^1 \otimes \mathfrak{p}) \times \mathcal{H}^{k,1}(\mathcal{Z}; \mathfrak{p}) \times \mathcal{H}^{k,2}(\mathcal{Z}; \text{Ad } P) \longrightarrow \mathcal{H}^{k,0}(\mathcal{Z}; \mathfrak{p}). \quad (\text{B.2})$$

with derivative

$$F := \partial_\gamma G(0, 0, 1) = d_A^* d_A - \text{ad } \Phi^2 : \mathcal{H}^{k,2}(\mathcal{Z}; \mathfrak{p}) \longrightarrow \mathcal{H}^{k,0}(\mathcal{Z}; \mathfrak{p}). \quad (\text{B.3})$$

Proof. Fix a fiber Z of $\mu : \mathcal{Z} \rightarrow \mathcal{I}$. That (B.2) is a bounded map over Z follows from Theorem 5.7, and the diagonality assumption on Φ , from which it follows that $\text{ad } \Phi$ maps $\mathcal{H}^{k,l}$ to $\mathcal{H}^{k,l} \cap \mathcal{H}^{k,0}$ for $l = 1, 2$ (while $\mathcal{H}^{k,2} \not\subset \mathcal{H}^{k,1} \not\subset \mathcal{H}^{k,0}$, $\text{ad } \Phi$ kills the \mathfrak{p}_0 components to infinite order). Since all the nonlinear terms are simple products, (B.2) is an analytic map.

Setting $\gamma = 1 + \eta$, where $\eta \in \mathcal{H}^{k,2}(Z; \mathfrak{p})$ and discarding terms of quadratic and higher order in η , we obtain the linearization

$$\partial_\gamma G(a, \phi, 1)\eta = d_A^* d_{A+a}\eta - \text{ad } \Phi \text{ad}(\Phi + \phi)\eta. \quad (\text{B.4})$$

Setting $(a, \phi) = (0, 0)$ gives (B.3). Letting Z vary, it is clear that, as bounded operators, G and F vary smoothly over \mathcal{I} . \square

We will show that (B.3) is invertible for sufficiently small ε and appeal to the implicit function theorem.

The restrictions of F in (B.3) to the boundary faces X and D of Z are analyzed in §C.3 where they are shown to be invertible, and in §D.6 we construct a smooth family of fiberwise parametrices for F on $\mu : \mathcal{Z} \rightarrow \mathcal{I}$. There we prove

Proposition B.2. *There exist right and left parametrices Q^R and Q^L for F such that, for some $0 < \delta < \frac{1}{2}$,*

$$FQ^R = I - \varepsilon^\delta E^R, \quad Q^L F = I - \varepsilon^\delta E^L, \quad (\text{B.5})$$

where E^R and E^L extend to (fiberwise bounded) linear maps on $\mathcal{H}^{k,0}(\mathcal{Z}; \mathfrak{p})$ and $\mathcal{H}^{k,2}(\mathcal{Z}; \mathfrak{p})$, respectively.

Proof of Theorem 5.8. Fixing a compact set $\mathcal{K} \subset \mathcal{I}$, and $\varepsilon_{\mathcal{K}} > 0$, the error terms in (B.5) satisfy bounds of the form $C\varepsilon_{\mathcal{K}}^\delta$ on $\mathcal{H}^{k,*}(\mathcal{Z}|_{\mathcal{K} \times [0, \varepsilon_{\mathcal{K}}]}; \mathfrak{p})$, and, making $\varepsilon_{\mathcal{K}}$ sufficiently small, can be inverted by Neumann series.

It follows that

$$F : \mathcal{H}^{k,2}(\mathcal{Z}|_{\mathcal{K} \times [0, \varepsilon_{\mathcal{K}}]}; \mathfrak{p}) \rightarrow \mathcal{H}^{k,0}(\mathcal{Z}|_{\mathcal{K} \times [0, \varepsilon_{\mathcal{K}}]}; \mathfrak{p})$$

is invertible map of Hilbert bundles over \mathcal{K} , and then the existence of a unique γ satisfying $G(a, \phi, \gamma) = 0$ for $(a, \phi) \in \mathcal{H}^{k,1}(\mathcal{Z}|_{\mathcal{K} \times [0, \varepsilon_{\mathcal{K}}]})$ sufficiently small (with respect to $\sup_{\mathcal{K}} \|\cdot\|_{\mathcal{H}^{k,1}}$) is a consequence of the implicit function theorem.

For the regularity statement it suffices to work on a fixed fiber Z . We proceed by induction on l , showing that there are unique solution maps

$$\begin{aligned} U_{l-1} &\subset \mathcal{H}^{k+l-1,1}(Z; \mathfrak{p}) \rightarrow \mathcal{H}^{k+l-1,2}(Z; \text{Ad } P) \\ U_l &= i^{-1}(U_{l-1}) \subset \mathcal{H}^{k+l,1}(Z; \mathfrak{p}) \rightarrow \mathcal{H}^{k+l,2}(Z; \text{Ad } P) \\ (a, \phi) &\mapsto \gamma \quad \text{s.t. } G(a, \phi, \gamma) = 0, \end{aligned} \quad (\text{B.6})$$

where the U_l are convex open neighborhoods of the origin and $i : \mathcal{H}^{k+l,1} \rightarrow \mathcal{H}^{k+l-1,1}$ denotes the natural inclusion. In particular, the domains don't decrease with l .

The above construction, applied with k and $k + 1$, furnishes the base case; shrinking U_0 if necessary, we may assume that $U_1 = i^{-1}(U_0)$. For the inductive step, suppose $(a, \phi) \in i^{-1}(U_l) \subset \mathcal{H}^{k+l+1,1}$ with the solution $\gamma = \exp(\eta)$, $\eta \in \mathcal{H}^{k+l,2}(Z; \mathfrak{p})$. Let $V \in \mathcal{V}_b(Z)$ be an arbitrary vector field, and consider

$$\begin{aligned} 0 &= V \cdot (G(a, \phi, \exp(\eta))) \\ &= (V \cdot G)(a, \phi, \exp(\eta)) + G_1(a, \phi, \exp(\eta))V(a, \phi) + G_2(a, \phi, \exp(\eta))V\eta. \end{aligned} \tag{B.7}$$

Here $G_1(a, \phi, \exp(\eta)) = \partial_{(a, \phi)} G(a, \phi, \exp(\eta))$ is the linearization of G with respect to the (a, ϕ) variables, $G_2(a, \phi, \exp(\eta)) = \partial_\eta G(a, \phi, \exp(\eta))$ is the linearization with respect to η , and $(V \cdot G)$ denotes all terms where V differentiates the coefficients of G , i.e., where V differentiates a term in the background configuration (A, Φ) .

By the smoothness assumption on (A, Φ) , the proof of Lemma B.1 applies to $(V \cdot G)$, and we conclude that

$$(V \cdot G) : \mathcal{H}^{k+l,1}(Z; \mathfrak{p}) \times \mathcal{H}^{k+l,2}(Z; \text{Ad } P) \longrightarrow \mathcal{H}^{k+l,0}(\mathfrak{p})$$

is a C^1 map. Likewise, the linear map

$$G_1(a, \phi, \exp(\eta)) : \mathcal{H}^{k+l,1}(Z; \mathfrak{p}) \longrightarrow \mathcal{H}^{k+l,0}(Z; \mathfrak{p})$$

is bounded, and $V(a, \phi)$ is in $\mathcal{H}^{k+l,1}$ by assumption. Finally, as a result of the inductive hypothesis (B.6), it follows that

$$\begin{aligned} G_2(a, \phi, \exp(\eta)) &: \mathcal{H}^{k+l-1,2}(Z; \mathfrak{p}) \longrightarrow \mathcal{H}^{k+l-1,0}(Z; \mathfrak{p}), \\ G_2(a, \phi, \exp(\eta)) &: \mathcal{H}^{k+l,2}(Z; \mathfrak{p}) \longrightarrow \mathcal{H}^{k+l,0}(Z; \mathfrak{p}) \end{aligned}$$

are isomorphisms, with inverses which coincide where defined. Rearranging (B.7), we conclude that

$$\begin{aligned} V\eta &= -G_2^{-1}(a, \phi, \exp(\eta))((V \cdot G)(a, \phi, \exp(\eta)) + G_1(a, \phi, \exp(\eta))V(a, \phi)) \\ &\in \mathcal{H}^{k+l,2}(Z; \mathfrak{p}). \end{aligned}$$

Since V was arbitrary, in fact $\eta \in \mathcal{H}^{k+l+1,2}$. Letting (a, ϕ) vary in $i^{-1}(U_l)$ and appealing to the uniqueness of the solution γ , we conclude that $(a, \phi) \longmapsto \gamma$ such that $G(a, \phi, \gamma) = 0$ defines a C^1 map

$$U_{l+1} := i^{-1}(U_l) \subset \mathcal{H}^{k+l+1,1}(Z; \mathfrak{p}) \longrightarrow \mathcal{H}^{k+l+1,2}(Z; \text{Ad } P),$$

which completes the induction. \square

APPENDIX C. LINEAR ANALYSIS

C.1. Linear analysis of L_{X_i} . In this section we give the analysis of the operator $L_{(A, \Phi)}$ (3.31) as needed for the construction of the formal solution in §3.7 over the Euclidean boundary hypersurfaces \mathcal{X}_i . Thus let X be the radial compactification of \mathbb{R}^3 , with boundary defining function ρ and let (A, Φ) be a smooth solution of the Bogomolny equations. This operator has

been considered previously by Taubes and more systematically by the first author. We need to refine the parametrix found by Kottke in [Kot15c] for applications in this paper; on the other hand there are some simplifications that result from X being the radial compactification of \mathbb{R}^3 rather than a general scattering manifold. Denote by Λ the bundle ${}^{\text{sc}}\Lambda^1 \oplus {}^{\text{sc}}\Lambda^0$ over \mathbb{R}^3 and by \mathfrak{p} the complexification of the adjoint bundle $\text{ad}(P)$. Then

$$L_X = L_{(A, \Phi)} \begin{bmatrix} a \\ \phi \end{bmatrix} = \begin{bmatrix} *d_A & -d_A \\ -d_A^* & 0 \end{bmatrix} \begin{bmatrix} a \\ \phi \end{bmatrix} + \text{ad}(\Phi) \begin{bmatrix} a \\ \phi \end{bmatrix}. \quad (\text{C.1})$$

where $(a, \phi) \in \Lambda \otimes \mathfrak{p}$.

We begin with the consequences of the decomposition $\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_1$ near ∂X . Let $\hat{\Phi} = \Phi/|\Phi|$.

Lemma C.1. *Let $C = \nabla_A \hat{\Phi}$, defined over a collar neighbourhood \mathcal{U} of ∂X . Then*

$$C \in \rho^\infty C^\infty(\mathcal{U}, \mathfrak{p}_1). \quad (\text{C.2})$$

Proof. By differentiation of $|\hat{\Phi}|^2 = 1$, we get $\langle \hat{\Phi}, C \rangle = 0$, showing that $C \in C^\infty(\mathcal{U}, \mathfrak{p}_1 \otimes {}^{\text{sc}}T^*)$. We also have

$$C = \nabla_A(|\Phi|^{-1}\Phi) = d(|\Phi|^{-1})\Phi + |\Phi|^{-1}\nabla_A\Phi \quad (\text{C.3})$$

Since also $\text{ad } \hat{\Phi} C = C$ by definition of \mathfrak{p}_1 , it follows that

$$C = |\Phi|^{-1} \text{ad}(\hat{\Phi}) \nabla_A \Phi \quad (\text{C.4})$$

and so the rapid decay of C follows from that of $\text{ad}(\Phi) \nabla_A \Phi$ discussed in §2. \square

Given any section u of \mathfrak{p} define $u_0 \in C^\infty(\mathcal{U}, \mathbb{C})$ by

$$u_0 = \langle \hat{\Phi}, u \rangle. \quad (\text{C.5})$$

Then

$$u_1 = u - u_0 \hat{\Phi} \quad (\text{C.6})$$

is a section of \mathfrak{p}_1 and thus satisfies

$$\text{ad } \hat{\Phi} u_1 = u_1. \quad (\text{C.7})$$

Denote by ∇_1 the connection on \mathfrak{p}_1 induced by projection of ∇_A on \mathfrak{p}_1 .

Proposition C.2. *Under the identification $\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_1$, $\mathfrak{p}_0 \simeq \mathbb{C}$ just described,*

$$\nabla_A \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} = \begin{bmatrix} du_0 \\ \nabla_1 u_1 \end{bmatrix} + \begin{bmatrix} 0 & \text{ad}(C) \\ C & 0 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \quad (\text{C.8})$$

Proof. If $u_0 \in C^\infty(\mathcal{U}, \mathbb{C})$, then we calculate

$$\nabla(u_0 \hat{\Phi}) = du_0 \otimes \hat{\Phi} + u_0 C. \quad (\text{C.9})$$

Thus the \mathfrak{p}_0 and \mathfrak{p}_1 components are precisely du_0 and $u_0 C$, proving the first line of (C.8). If $u_1 \in C^\infty(\mathcal{U}, \mathfrak{p}_1)$, then by definition

$$\nabla_1 u_1 = \text{ad}(\hat{\Phi}) \nabla_A u_1. \quad (\text{C.10})$$

Differentiating the equation $u_1 = \text{ad}(\hat{\Phi})u_1$, we obtain

$$\nabla_A u_1 = \text{ad}(\hat{\Phi})u_1 + \text{ad}(C)u_1 = \nabla_1 u_1 + \text{ad}(C)u_1. \quad (\text{C.11})$$

The result is proved. \square

As a consequence, we have (c.f. (3.32))

Proposition C.3. *Over \mathcal{U} , relative to the decomposition $\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_1$,*

$$L_X = \begin{bmatrix} L & C_1 \\ C_1^* & L_1 + \Phi_1 \end{bmatrix} \quad (\text{C.12})$$

where

$$L = \begin{bmatrix} *d & -d \\ -d^* & 0 \end{bmatrix}, \quad (\text{C.13})$$

is the euclidean Hodge-de Rham operator on \mathbb{R}^3 , L_1 is the same operator coupled to \mathfrak{p}_1 , Φ_1 denotes $\text{ad}(\Phi)$ acting on \mathfrak{p}_1 and C_1 is a zeroth order $O(\rho^\infty)$ term.

Thus, near \mathcal{U} , L_X behaves like a sum of the uncoupled euclidean Hodge-de Rham operator L and the fully elliptic scattering operator $L_1 + \Phi_1$.

As is well known, Fredholm extensions, solvability and boundary regularity properties of the \mathfrak{p}_0 component of L_X are therefore governed by the homogeneous solutions of $Lu = 0$.

Proposition C.4. *Suppose that $Lu = 0$ over $\mathbb{R}^3 \setminus 0$ and u is homogeneous of degree α . Then if $\alpha \geq 0$, it follows that $\alpha = n$ is a non-negative integer, and there is a homogenous harmonic polynomial h of degree $n+1$, such that*

$$u = Lh \quad (\text{C.14})$$

If $\alpha < 0$ then $\alpha = -2 - n$, where n is a non-negative integer, and there is a harmonic polynomial, homogeneous of degree n , such that

$$u = L(|z|^{-2n-1}h) \quad (\text{C.15})$$

Proof. It is clear that (C.14) and (C.15) do give homogenous solutions of the given degree, because $L^2 = \Delta$. Let us consider (C.15). If $Lu = 0$ and u is homogeneous of some negative degree α , then Lu extends to \mathbb{R}^3 uniquely as a homogeneous distribution supported at 0. The only possibility is a linear combination of derivatives of the Dirac distribution δ_0 . Since this distribution is of degree -3 , it follows that $s = Lu$ can only be homogeneous of degree $-3 - n$, for some $n \geq 0$. Applying L , $\Delta u = Ls$ and $u = Lf$, where f is the unique homogeneous solution of $\Delta f = s$. Thus f has the form $|z|^{-2n-1}h$ where h is an \mathbb{R}^4 -valued harmonic polynomial, homogeneous of degree n , proving the result.

The proof of (C.15) is similar; by elliptic regularity, if the homogeneity is non-negative, then $Lu = 0$ on \mathbb{R}^3 . So u is an \mathbb{R}^4 -valued harmonic polynomial, homogeneous of degree $n \geq 0$. However any such polynomial can be written

$u = \Delta v$ (take v a multiple of $|z|^2 u$). Then $u = L(Lv)$ and Lv is an \mathbb{R}^4 -valued function, homogeneous of degree $n + 1$. \square

The remainder of the section is devoted to a proof of the following

Theorem C.5.

- (a) For $m \geq 0$, $\beta \in \mathbb{R}$, and $\alpha \in (-1, 1)$ the bounded extension

$$L_X : \rho^{\alpha-1/2, \beta} H_{\text{b/sc}}^{m,1}(X; \Lambda \otimes \mathfrak{p}) \longrightarrow \rho^{\alpha+1/2, \beta} H_{\text{b/sc}}^{m,0}(X; \Lambda \otimes \mathfrak{p}) \quad (\text{C.16})$$

is Fredholm and surjective. The spaces here are defined in §A.1. The null-space N is of complex dimension $4k$ and if $u \in N$ is decomposed as $u = u_0 + u_1$ relative to $\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_1$ near the boundary,

$$u_0 \in \rho^2 C^\infty(\mathcal{U}), u_1 \in \rho^\infty C^\infty(\mathcal{U}, \mathfrak{p}_1). \quad (\text{C.17})$$

- (b) There is a right-inverse G of (C.16) with range equal to the L^2 orthogonal complement of N with the following property. If $f \in \mathcal{A}^*(X; \Lambda \otimes \mathfrak{p})$ with $f = f_0 \oplus f_1$ near ∂X ,

$$f_i \in \mathcal{A}^{F_i}(X; \Lambda \otimes \mathfrak{p}_i), \quad (\text{C.18})$$

then $u = Gf$ solves $Lu = f$ with $u \in \mathcal{A}^*(X; \Lambda \otimes \mathfrak{p})$. Moreover, decomposing $u = u_0 + u_1$ near ∂X , we have $u_i \in \mathcal{A}^{E_i}$, where

$$\begin{aligned} E_0 &= \widehat{2\mathbb{U}}(F_0 - 1), \\ E_1 &= F_1. \end{aligned}$$

- (c) Let \mathcal{U} be a product neighbourhood of ∂X as before and consider the Schwarz kernel of G on restricted to $\mathcal{U} \times \mathcal{U}$. Then $G|_{\mathcal{U} \times \mathcal{U}}$ decomposes with respect to $\text{Hom}(\pi_R^*(\mathfrak{p}_0 \oplus \mathfrak{p}_1), \pi_L^*(\mathfrak{p}_0 \oplus \mathfrak{p}_1))$

$$G = \begin{pmatrix} \rho \tilde{G}_0 \rho^{-2} & G_{01} \\ G_{10} & G_1 \end{pmatrix}, \quad (\text{C.19})$$

$$\tilde{G}_0 \in \Psi_{\text{b}}^{-1, \mathcal{F}}(X; \Lambda \otimes \mathfrak{p}_0), \quad \mathcal{F} = (F_L, F_R, F_F), \quad F_L, F_R, F_F \geq 0.$$

$$G_1 \in \Psi_{\text{sc}}^{-1,0}(X; \Lambda \otimes \mathfrak{p}_1), \quad G_{ij} \in \rho^\infty \Psi^{-1}(X; \Lambda \otimes \mathfrak{p}_j, \Lambda \otimes \mathfrak{p}_i).$$

Here $\rho^\infty \Psi^{-1}$ is well-defined in either calculus as $\rho^\infty \Psi^{-1} = \Psi_{\text{b}}^{-1, (\infty, \infty, \infty)} = \Psi_{\text{sc}}^\infty$.

The proof of this is an elaboration of work of the first author in [Kot15c] and starts from a parametrix construction using both the b and sc calculi as well as the pseudo-differential operators in $\rho^\infty \Psi^s(X)$. Note that this forms a bi-ideal in either calculus, in the sense that

$$\begin{aligned} \rho^\infty \Psi^k \circ \Psi_{\text{b}}^{l, \mathcal{E}} &\subset \rho^\infty \Psi^{k+l}, & \Psi_{\text{b}}^{l, \mathcal{E}} \circ \rho^\infty \Psi^k &\subset \rho^\infty \Psi^{k+l}, \\ \rho^\infty \Psi^k \circ \Psi_{\text{sc}}^{l, e} &\subset \rho^\infty \Psi^{k+l}, & \Psi_{\text{sc}}^{l, e} \circ \rho^\infty \Psi^k &\subset \rho^\infty \Psi^{k+l}. \end{aligned} \quad (\text{C.20})$$

It is convenient to replace L by $\tilde{L} = \rho^{-2} L \rho$ in what follows. We also omit mention of Λ since this is a passenger and the important thing is the

splitting $\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_1$. In order to construct a parametrix near the boundary for the \mathfrak{p}_0 part \tilde{L} , we need to know its indicial roots:

Lemma C.6. $\text{spec}_b(\tilde{L}) = \pm\mathbb{N}_1 = \mathbb{Z} \setminus \{0\}$.

Proof. This follows at once from Proposition C.4 and the definition $\tilde{L} = \rho^{-2}L\rho$. The calculation is done in the more general setting that X is an arbitrary scattering 3-manifold in [Kot15c], by identifying L with the odd-signature operator on X . \square

As an initial step, let Q be a distribution on X^2 conormal to the diagonal on the interior, with principal symbol inverting that of L_X , and decomposing near \mathcal{U}^2 as

$$Q = \begin{pmatrix} \rho\tilde{Q}_0\rho^{-2} & 0 \\ 0 & Q_1 \end{pmatrix}.$$

Here we assume that $Q_1 \in \Psi_{\text{sc}}^{-1,0}(X; \mathfrak{p}_1)$ has scattering symbol inverting that of $L_1 + \Phi_1$ (which is invertible by the fact that L_1 is self-adjoint, Φ_1 is skew-adjoint and nondegenerate on \mathfrak{p}_1 , and these commute to leading order at ∂X), and we assume that $\tilde{Q}_0 \in \Psi_b^{-1,(\hat{1}_L, 1_R, 0_F)}(X; \mathfrak{p}_0)$ satisfies $I - \tilde{L}\tilde{Q}_0 \in \Psi_b^{-1,(\infty_L, 1_R, 1_F)}(X; \mathfrak{p}_0)$, following the first few standard steps in the construction of parametrices in the b-calculus [Mel93]. More precisely we assume that \tilde{Q}_0 has interior principal symbol inverting that of \tilde{L} , that the indicial operator $I(\tilde{Q}_0)$ is obtained by taking the inverse Mellin transform of $I(\tilde{L}, \lambda)$ along $\text{Re}(\lambda) = \alpha \in (-1, 1)$ (which is free of indicial roots by Proposition C.6; the indicial roots to the right and left of this line contribute the index set 1 at the right and left faces for \tilde{Q}_0), and that the Schwartz kernel of \tilde{Q}_0 is in the formal nullspace of the lift of \tilde{L} to X_b^2 at the left face, contributing the index set $\hat{1}$ for \tilde{Q}_0 and the rapid vanishing of $I - \tilde{L}\tilde{Q}_0$ there. These assumptions are all consistent with the choice of interior conormal symbol of Q .

It follows that the initial error term $E = I - L_X Q$ has the form

$$\begin{aligned} E &= \begin{pmatrix} \rho^2 \tilde{E}_0 \rho^{-2} & E_{01} \\ E_{10} & E_1 \end{pmatrix}, \\ \tilde{E}_0 &\in \Psi_b^{-1,(\infty, 1, 1)}(X; \mathfrak{p}_0), \quad E_1 \in \Psi_{\text{sc}}^{-1,1}(X; \mathfrak{p}_1), \\ E_{01} &= -C_1 Q_1 \in \rho^\infty \Psi^{-1}(X; \mathfrak{p}_1, \mathfrak{p}_0) \\ E_{10} &= -C_1^* \rho \tilde{Q}_0 \rho^{-2} \in \rho^\infty \Psi^{-1}(X; \mathfrak{p}_0, \mathfrak{p}_1), \end{aligned}$$

near $(\partial X)^2$, with interior conormal singularity of order -1 . This term may now be removed by Neumann series. Indeed, absorbing terms $\rho^\infty \Psi^{-N}$ into

$\Psi_b^{-N,(\infty,\infty,\infty)}$ and $\Psi_{sc}^{-N,\infty}$, it follows that

$$E^N = \begin{pmatrix} \rho^2 \tilde{E}_0^{(N)} \rho^{-2} & E_{01}^{(N)} \\ E_{10}^{(N)} & E_1^{(N)} \end{pmatrix},$$

$$\tilde{E}_0^{(N)} \in \Psi_b^{-N,(\infty,\hat{1}_N,N)}(X; \mathfrak{p}_0), \quad E_1^{(N)} \in \Psi_{sc}^{-N,N}(X; \mathfrak{p}_1),$$

$$E_{ij}^{(N)} \in \rho^\infty \Psi^{-N}$$

where $\hat{1}_N = 1 \sqcup 2 \sqcup \dots \sqcup N$. The series $\sum_{k=0}^\infty E^k$ may be summed asymptotically, resulting in an operator $I - S$, where S has an expression similar to E , except that $\tilde{S}_0 \in \Psi_b^{-1,(\infty,\hat{1},1)}(X; \mathfrak{p}_0)$. Denoting an improved right parametrix by $Q^R = Q(I - S)$, it follows that the new error term $E^R = I - L_X Q^R$ has the form

$$E^R = \begin{pmatrix} \rho^2 \tilde{E}_0^R \rho^{-2} & E_{01}^R \\ E_{10}^R & E_1^R \end{pmatrix},$$

$$\tilde{E}_0^R \in \Psi_b^{-\infty,(\infty,\hat{1},\infty)}(X; \mathfrak{p}_0), \quad E_1^R \in \Psi_{sc}^{-\infty,\infty}(X; \mathfrak{p}_1),$$

$$E_{ij}^R \in \rho^\infty \Psi^{-\infty}$$

The parametrix Q^R has a block form similar to the block form of Q , with $\tilde{Q}_0^R \in \Psi_b^{-1,(\hat{1},\hat{1} \sqcup 2, \hat{2} \sqcup 1 \sqcup 0)}(X; \mathfrak{p}_0)$, and off-diagonal terms in $\rho^\infty \Psi^{-1}$. The precise form of the index sets is not as important as the statement that \tilde{Q}_0^R lies in the b-calculus.

A similar construction gives a left parametrix Q^L with error term $E^L = I - Q^L L_X$ of the form

$$E^L = \begin{pmatrix} \rho \tilde{E}_0^L \rho^{-1} & E_{01}^L \\ E_{10}^L & E_1^L \end{pmatrix},$$

$$\tilde{E}_0^L \in \Psi_b^{-\infty,(\hat{1},\infty,\infty)}(X; \mathfrak{p}_0), \quad E_1^L \in \Psi_{sc}^{-\infty,\infty}(X; \mathfrak{p}_1),$$

$$E_{ij}^L \in \rho^\infty \Psi^{-\infty}$$

With these parametrices in hand, we may prove the following:

Proposition C.7.

- (a) *The extension (C.16) is Fredholm and surjective for $\alpha \in (-1, 1)$, $m \geq 0$ and $\beta \in \mathbb{R}$.*
- (b) *The nullspace of such an extension consists of polyhomogeneous sections, with*

$$\text{Null}(L_X) \ni u = u_0 \oplus u_1, \quad \text{near } \partial X,$$

$$u_0 \in \rho^2 C^\infty(X; \mathfrak{p}_0), \quad u_1 \in \rho^\infty C^\infty(X; \mathfrak{p}_1).$$

Proof. To see that the extension is Fredholm, it suffices to verify that E^L and E^R are compact on the appropriate spaces. Over the interior of X this

is clear, and near the boundary we have

$$\begin{aligned}\rho \tilde{E}_0^L \rho^{-1} &: \rho^{\alpha-1/2} H_b^{m+1} \longrightarrow \rho^{\alpha+1/2} H_b^\infty \subset \rho^{\alpha-1/2} H_b^{m+1}, \\ E_1^L &: \rho^\beta H_{b,sc}^{m,1} \longrightarrow \rho^\infty H_{b,sc}^{\infty,1} \subset \rho^\beta H_{b,sc}^{m,1} \\ E_{10} &: \rho^{\alpha-1/2} H_b^{m+1} \longrightarrow \rho^\infty H_b^\infty \subset \rho^\beta H_{b,sc}^{m,1} \\ E_{01} &: \rho^\beta H_{b,sc}^{m,1} \longrightarrow \rho^\infty H_{b,sc}^{\infty,1} \subset \rho^{\alpha-1/2} H_b^{m+1}\end{aligned}$$

where all inclusions are compact by Proposition A.2. The argument for E^R is similar.

For (b), observe that E^L maps $\rho^{\alpha-1/2,\beta} H_{b/sc}^{k,1}$ into $\rho^{0,\infty} \mathcal{A}^{\hat{2}}$, meaning polyhomogeneous sections whose \mathfrak{p}_1 components are rapidly vanishing and whose \mathfrak{p}_0 components have index set $\hat{2} = \hat{1} + 1$. Thus, supposing $L_X u = 0$ and applying the left parametrix, it follows that $u = E^L u \in \rho^{0,\infty} \mathcal{A}^{\hat{2}}$.

In fact, since X is Euclidean, $\tilde{L} \equiv I(\tilde{L})$ agrees identically with its indicial operator in a neighborhood of ∂X . It then follows from $L_X u = 0$ that $I(\tilde{L})\rho^{-1}u_0 = 0 \pmod{\rho^\infty C^\infty}$, and so $u_0 \in \rho^2 C^\infty$.

Surjectivity will follow from injectivity for the adjoint operator

$$L_X^* : \rho^{-\alpha-1/2,-\beta} H_{b/sc}^{m,1} \longrightarrow \rho^{-\alpha+1/2,-\beta} H_{b/sc}^{m,0}. \quad (\text{C.21})$$

Here we are considering the adjoint determined by the L^2 pairing between weighted spaces $\rho^{\alpha,\beta} L^2$ and $\rho^{-\alpha,-\beta} L^2$, with $H_{b/sc}^{m,l}$ functioning as domains for the unbounded operators L_X and L_X^* . Note that $-\alpha \in (-1,1)$ since $\alpha \in (-1,1)$ by assumption. Since $L_X^* = L_A - \text{ad}(\Phi)$ differs from L_X only in the sign of the zeroth order term, we may apply the foregoing analysis to it, and in particular deduce that if u is in the null space of (C.21), then it is $O(\rho^2)$ (and conormal).

Now we use the basic identity

$$L_X L_X^* = \nabla_A^* \nabla_A - (\text{ad}(\Phi))^2 \quad (\text{C.22})$$

which follows from the Bogomolny equations and the fact that \mathbb{R}^3 is (Ricci) flat. If $L_X^* u = 0$, $u = O(\rho^2)$, we have

$$(u, L_X L_X^* u) = (u, \nabla_A^* \nabla_A u) + \|\text{ad}(\Phi)u\|^2 \quad (\text{C.23})$$

as u and its derivatives are all in L^2 . The decay at ∂X is sufficient to integrate by parts with no boundary term, we conclude

$$0 = \|L_X^* u\|^2 = \|\nabla_A u\|^2 + \|\text{ad}(\Phi)u\|^2 \quad (\text{C.24})$$

from which u is covariant constant hence zero because vanishing at the boundary. \square

It follows from the previous result that there exists a bounded right inverse to L_X on the Sobolev spaces in consideration:

$$\begin{aligned}G : \rho^{\alpha+1/2,\beta} H_{b/sc}^{m,0}(X; \mathfrak{p}) &\longrightarrow \rho^{\alpha-1/2,\beta} H_{b/sc}^{m,1}(X; \mathfrak{p}), \\ L_X G &= I, \quad G L_X = I - \Pi_N\end{aligned}$$

where Π_N is projection on the null space N of L_X . From the identities

$$\begin{aligned} Q^L &= Q^L L_X G = G - E^L G, \\ G - G E^R &= G L_X Q^R = Q^R - \Pi_N Q^R, \end{aligned}$$

it follows that G satisfies

$$G = Q^R + Q^L E^R - \Pi_N Q^R + E^L G E^R.$$

Because of the bi-ideal properties of $\rho^\infty \Psi^{-\infty} = \Psi_{\text{sc}}^{-\infty, \infty}$ and left (resp. right) ideal properties of $\Psi_{\text{b}}^{-\infty, (\infty, *, \infty)}$ (resp. $\Psi_{\text{b}}^{-\infty, (*, \infty, \infty)}$), the last term has the form

$$E^L G E^R \in \begin{pmatrix} \rho \Psi_{\text{b}}^{-\infty, (\hat{1}, \hat{1}, \infty)} \rho^{-2} & \rho^\infty \Psi^{-\infty} \\ \rho^\infty \Psi^{-\infty} & \Psi_{\text{sc}}^{-\infty, \infty} \end{pmatrix}$$

near $(\partial X)^2$. Computing the compositions of the other terms leads to a proof of Theorem C.5.(c). Note that the precise index sets are not of critical importance; for the present purpose it suffices to only keep track of the leading orders.

As a consequence, G maps polyhomogeneous sections to polyhomogeneous sections:

$$G : \mathcal{A}^* \cap \rho^{\alpha+1/2, \beta} H_{\text{b/sc}}^{k, 0}(X; \mathfrak{p}) \longrightarrow \mathcal{A}^*(X; \mathfrak{p}). \quad (\text{C.25})$$

The precise behavior of the index sets, as in Theorem C.5.(b), is then determined *a posteriori*, rather than from the estimates on the index sets for G .

The following is a general result for b differential operators:

Proposition C.8 ([Mel93], Prop. 5.61, p. 205). *Let $P \in \text{Diff}_{\text{b}}^k(X; V)$, and suppose $u \in \mathcal{A}^G(X; V) \cap \rho^\alpha H_{\text{b}}^k(X; V)$ satisfies $Pu = f \in \mathcal{A}^F(X; V) \cap x^\alpha L^2(X; V)$, where $\alpha \notin \text{spec}_{\text{b}}(P)$. Then in fact $u \in \mathcal{A}^E(X; V)$, where*

$$E = \hat{E}^+(\alpha) \sqcup F, \quad (\text{C.26})$$

where $E^+(\alpha) = \{(z, k) \in \text{Spec}_{\text{b}}(P) : \text{Re}(z) > \alpha\}$.

Proof of Theorem C.5. Parts (a) and (c) were proved above; it remains to prove part (b). Suppose $f = f_0 \oplus f_1$ near ∂X with $f_0 \in \mathcal{A}^{F_0}(X; \mathfrak{p}_0)$, $f_1 \in \mathcal{A}^{F_1}(X; \mathfrak{p}_1)$ and let $u = Gf$, which is polyhomogeneous by (C.25). Thus $L_X u = f$ and $u = u_0 \oplus u_1$ near ∂X with $u_i \in \mathcal{A}^{G_i}$, for some index sets G_0 and G_1 . We now use the fact that L_X has the form

$$L_X = \begin{pmatrix} \rho^2 \tilde{D}_0 \rho^{-1} & 0 \\ 0 & D_1 + \text{ad } \Phi \end{pmatrix} + \mathcal{O}(\rho^\infty).$$

with $\tilde{D}_0 \in \text{Diff}_{\text{b}}^1(X; \mathfrak{p}_0)$. Since the off-diagonal terms coupling between \mathfrak{p}_0 and \mathfrak{p}_1 are vanishing rapidly, they have no effect on the terms which actually appear in the index sets and will be ignored.

It suffices therefore to suppose that, near ∂X , we have

$$\begin{aligned}\rho^2 \tilde{D}_0 \rho^{-1} u_0 &= f_0 \in \mathcal{A}^{F_0}(X; \mathfrak{p}_0), \\ f D_1 u_1 + \text{ad } \Phi u_1 &= f_1 \in \mathcal{A}^{F_1}(X; \mathfrak{p}_1).\end{aligned}$$

Rewriting the first equation as $\tilde{D}_0(\rho^{-1} u_0) = \rho^{-2} f_0 \in \mathcal{A}^{F_0-2}$ and invoking Proposition C.8 with $E^+(\alpha) = 1$, we obtain

$$\begin{aligned}u_0 &\in \mathcal{A}^{E_0}(X; \mathfrak{p}_0), \\ E_0 &= \widehat{1}\overline{\cup}(F_0 - 2) + 1 = \widehat{2}\overline{\cup}(F_0 - 1)\end{aligned}$$

as claimed.

In the second equation, $\text{ad } \Phi$ is the dominant term in the operator as far as polyhomogeneity is concerned; thus we may write

$$\text{ad } \Phi u_1 = f_1 - \rho \tilde{D}_1 u_1 \quad \tilde{D}_1 \in \text{Diff}_b^1(X; \mathfrak{p}_1).$$

Proceeding inductively over the leading terms in the index set F_1 , we conclude that $u_1 \in \mathcal{A}^{F_1}(X; \mathfrak{p}_1)$. \square

C.2. Linear analysis of L_D . Next we consider the linear operator

$$L_D = \begin{bmatrix} *d & -d \\ -d^* & 0 \end{bmatrix}$$

on $\Lambda = \Lambda^1 \oplus \Lambda^0$ the face D .

Theorem C.9.

(a) For $\alpha, \beta \in (-1, 1)$ and any $k \geq 0$,

$$L_D : \rho_B^{\alpha-1/2} \rho_X^{\beta+1/2} H_b^k(D; \Lambda) \longrightarrow \rho_B^{\alpha+1/2} \rho_X^{\beta-1/2} H_b^{k-1}(D; \Lambda) \quad (\text{C.27})$$

is invertible. Here the Sobolev spaces are based on $L^2(D)$ with respect to the rescaled metric $\varepsilon^{-2}g|_D$ of conic/scattering type and $\rho_X = \prod_i \rho_{X_i}$.

(b) Given

$$\begin{aligned}f &\in \mathcal{A}^{\mathcal{F}}(D; \Lambda) \cap \rho_B^{\alpha+1/2} \rho_X^{\beta-1/2} H_b^{k-1}(D; \Lambda), \\ \mathcal{F} &= (F_B, F_X),\end{aligned}$$

there exists a unique $u \in \mathcal{A}^{\mathcal{E}}(D; \Lambda)$ such that $L_D u = f$, where

$$\begin{aligned}\mathcal{E} &= (E_B, E_X), \\ E_B &= \widehat{2}\overline{\cup}(F_B - 1), \\ E_X &= \widehat{0}\overline{\cup}(F_X + 1).\end{aligned}$$

(c) The inverse L_D^{-1} to (C.27) has the form

$$\begin{aligned}L_D^{-1} &= \rho_B \rho_X^{-1} \tilde{G} \rho_X^2 \rho_B^{-2}, \quad \tilde{G} \in \Psi_b^{-1, \mathcal{F}}(D; \Lambda), \\ \mathcal{F} &= (F_L, F_R, F_F), \quad F_L, F_R \geq 1, \quad F_F \geq 0.\end{aligned} \quad (\text{C.28})$$

The proof is similar to the proof of Theorem C.5, so we shall be brief.

Recall that, as a compact manifold with boundary, $D = [X; \{\zeta_j\}]$ is the radial compactification of euclidean \mathbb{R}^3 , blown up at a finite number of points.

Thus, as L_D is a homogeneous operator of order 1, of conic type near $D \cap X_i$, and scattering type near $D \cap B$, the first step in the proof of Theorem C.9 is to define the related b operator

$$\tilde{L} = \rho_B^{-2} \rho_X^2 L_D \rho_X^{-1} \rho_B \in \text{Diff}_b^1(D; \Lambda). \quad (\text{C.29})$$

Observe that the mapping properties (boundedness, invertibility, Fredholmness, self-adjointness, etc.) of

$$\tilde{L} : \rho_B^\alpha \rho_X^\beta H_b^k(D; \Lambda; g_b) \longrightarrow \rho_B^\alpha \rho_X^\beta H_b^{k-1}(D; \Lambda; g_b) \quad (\text{C.30})$$

are the same as the mapping properties of (C.27), where $g_b = \rho_B^2 \rho_X^{-2} \varepsilon^{-2} g$ is the conformally related b-metric.

Proposition C.10. *For $\alpha, \beta \in (-1, 1)$, the extension (C.27) is invertible, and $\text{Null}(L_D) \in \rho_B^2 C^\infty(D; \Lambda)$*

Proof. Proceeding as in Proposition C.6, we see that the indicial roots at all boundary faces of D are given by

$$\text{spec}_b(\tilde{L}) = \mathbb{Z} \setminus \{0\}.$$

As the extension (C.27) of L_D and hence the extension (C.30) \tilde{L} are self-adjoint when $\alpha = \beta = 0$, it follows from standard results for b differential operators that (C.27) is Fredholm with index 0 for $\alpha, \beta \in (-1, 1)$, and it follows from the fact that \tilde{L} agrees identically with its indicial operator near X_i and B that $\text{Null}(L_D) \in \rho_B^2 C^\infty(D; \Lambda)$.

To see that (C.27) is invertible, we use the Bochner formula

$$L_D^* L_D = \Delta = \nabla^* \nabla + \text{Ric} = \nabla^* \nabla,$$

where the adjoints are computed with respect to $L^2(D; \Lambda; \varepsilon^{-2} g)$. This formula follows as in the proof of Proposition C.7. Thus for $u \in \text{Null}(L_D)$ with respect to any extension (C.27) with $\alpha, \beta \in (-1, 1)$,

$$\|\nabla u\|_{L^2}^2 = \langle \nabla^* \nabla u, u \rangle = \langle L_D^* L_D u, u \rangle = 0$$

and since $u|_B = 0$, u must vanish identically. The integration by parts is justified by comparing the decay rate $\mathcal{O}(\rho_B^2 \rho_X^0)$ of the nullspace with the volume element near B and X . It follows that (C.27) is injective, and therefore surjective since it has index 0. \square

Proof of Theorem C.9. Part (a) has been shown, and part (b) then follows immediately from Proposition C.8 and (C.29). Finally, part (c) follows by expressing the inverse, $\tilde{G} = \tilde{L}^{-1}$, of (C.30) as

$$\tilde{G} = \tilde{Q}_R + \tilde{Q}_L \tilde{E}_R + \tilde{E}_L \tilde{G} \tilde{E}_R \in \Psi_b^{-1, \mathcal{F}}(D; \Lambda),$$

where $\tilde{Q}_R \in \Psi_b^{-1,(\hat{1}\widetilde{\cup}2,\hat{1},\hat{2}\widetilde{\cup}1\cup 0)}(D; \Lambda)$ is a right parametrix for \tilde{L} with $\tilde{E}_R = I - \tilde{Q}_R \tilde{P}_0 \in \Psi_b^{-\infty,(\hat{1},\infty,\infty)}(D; \Lambda)$, and likewise $\tilde{Q}_L \in \Psi_b^{-1,(\hat{1},\hat{1}\widetilde{\cup}2,\hat{2}\widetilde{\cup}1\cup 0)}(D; \Lambda)$ is a left parametrix with $\tilde{E}_L = I - \tilde{P}_0 \tilde{Q}_L \in \Psi_b^{-\infty,(\infty,\hat{1},\infty)}(D; \Lambda)$. These may be constructed as in the previous section. \square

C.3. Linear analysis for Coulomb gauge. Here we analyze the linearized operator

$$F = \Delta_A - \text{ad } \Phi^2 = d_A^* d_A - \text{ad } \Phi^2$$

from the Coulomb gauge fixing problem. Here (A, Φ) are assumed to be smooth and diagonal to infinite order near $\mathcal{D} \cap \mathcal{B}$, and we restrict our analysis to boundary faces X and D of a fiber Z of \mathcal{Z} over \mathcal{I} .

Over X , we may assume A is in radial gauge with respect to the boundary defining function ρ for $X \cap D$. It follows that

$$F_X := N_{\mathcal{X}}(F)|_X = \begin{pmatrix} \Delta_A & F_{01} \\ F_{10} & \Delta_A - \text{ad } \Phi^2 \end{pmatrix} \quad F_{ij} \in \rho^\infty \Psi^1(X; \mathfrak{p})$$

On \mathfrak{p}_1 , the term $-\text{ad } \Phi^2 = (\text{ad } \Phi)^*(\text{ad } \Phi)$ is positive and nondegenerate so that $\Delta_A - \text{ad } \Phi^2 \in \text{Diff}_{\text{sc}}^1(X; \mathfrak{p}_1)$ is fully elliptic as a scattering operator [Mel94]. On \mathfrak{p}_0 we write

$$\begin{aligned} \Delta_A &= \rho^{1+3/2} \tilde{\Delta}_A \rho^{1-3/2} = \rho^{5/2} \tilde{\Delta}_A \rho^{-1/2} \\ \tilde{\Delta}_A &= -(\rho \partial_\rho)^2 + \frac{1}{4} + \Delta_{A, \partial X} \in \text{Diff}_b^1(X; \mathfrak{p}_0). \end{aligned}$$

Here $\Delta_{A, \partial X}$ denotes the scalar Laplacian on $\partial X = \mathbb{S}^2$ determined by the restriction of A (which is well-defined as A is a true connection) to ∂X and the rescaled metric $\rho^2 g|_{\partial X} = h_{\mathbb{S}^2}$. This explicit form of $\tilde{\Delta}_A$ follows by a direct computation from the assumption that A is in radial gauge. As a consequence,

$$\text{spec}_b(\tilde{\Delta}_A) = \left\{ \pm \sqrt{\frac{1}{4} + \nu} : \nu \in \text{spec}(\Delta_{A, \partial X}) \right\}.$$

In particular the interval $(-\frac{1}{2}, \frac{1}{2})$ is disjoint from $\text{spec}_b(\tilde{\Delta}_A)$.

Theorem C.11. *For $\alpha \in (-\frac{1}{2}, \frac{1}{2})$, and any $\beta \in \mathbb{R}$, $k \in \mathbb{N}$, the bounded operator*

$$F_X : \rho^{\alpha-1, \beta} H_{b/\text{sc}}^{k, 2}(X; \mathfrak{p}) \longrightarrow \rho^{\alpha+1, \beta} H_{b/\text{sc}}^{k, 0}(X; \mathfrak{p}) \quad (\text{C.31})$$

is invertible, with inverse G represented as a conormal distribution on X^2 , decomposing with respect to $\text{Hom}(\pi_R^(\mathfrak{p}_0 \oplus \mathfrak{p}_1), \pi_L^*(\mathfrak{p}_0 \oplus \mathfrak{p}_1))$ as*

$$\begin{aligned} G &= \begin{pmatrix} \rho^{1/2} \tilde{G}_0 \rho^{-5/2} & G_{01} \\ G_{10} & G_1 \end{pmatrix} \\ \tilde{G}_0 &\in \Psi_b^{-2, \mathcal{F}}(X; \mathfrak{p}_0) \quad \mathcal{F} = (F_L, F_R, F_F), \quad F_L, F_R \geq \frac{1}{2}, \quad F_F \geq 0. \\ G_1 &\in \Psi_{\text{sc}}^{-2, 0}(X; \mathfrak{p}_1), \quad G_{ij} \in \rho^\infty \Psi^{-2}(X; \mathfrak{p}_j, \mathfrak{p}_i) \end{aligned} \quad (\text{C.32})$$

Proof. The construction is similar to the one in Theorem C.5, so we shall be brief. Fixing the parameters $\alpha \in (-\frac{1}{2}, \frac{1}{2})$, $\beta \in \mathbb{R}$, and $k \in \mathbb{N}$, we proceed as in §C.1, beginning with an initial right parametrix Q of the form

$$Q = \begin{pmatrix} \rho^{1/2} \tilde{Q}_0 \rho^{-5/2} & 0 \\ 0 & Q_1 \end{pmatrix}$$

$$\tilde{Q}_0 \in \Psi_b^{-2, (\hat{S}, S, 0)}(X; \mathfrak{p}_0), \quad Q_1 \in \Psi_{sc}^{-2, 0}(X; \mathfrak{p}_1),$$

where $S = \{(s, k) \in \text{Spec}_b(\tilde{\Delta}_A) : s \geq 1/2\}$ and $\hat{S} = \bigcup_{n \in \mathbb{N}} (S + n)$, for which the error $E = I - F_X Q$ has the form

$$E = \begin{pmatrix} \rho^{5/2} \tilde{E}_0 \rho^{-5/2} & E_{01} \\ E_{10} & E_1 \end{pmatrix}$$

$$\tilde{E}_0 \in \Psi_b^{-1, (\infty, S, 1)}(X; \mathfrak{p}_0), \quad E_1 \in \Psi_{sc}^{-1, 1}(X; \mathfrak{p}_1),$$

$$E_{ij} \in \rho^\infty \Psi^{-1}$$

Summing the Neumann series for E leads to the improved right parametrix $Q^R = Q(\sum_{N=0}^\infty E^N)$ with $E^R = I - F_X Q^R$ of the form

$$E^R = \begin{pmatrix} \rho^{5/2} \tilde{E}_0^R \rho^{-5/2} & E_{01}^R \\ E_{10}^R & E_1^R \end{pmatrix}$$

$$\tilde{E}_0^R \in \Psi_b^{-\infty, (\infty, \hat{S}, \infty)}(X; \mathfrak{p}_0), \quad E_1^R \in \Psi_{sc}^{-\infty, \infty}(X; \mathfrak{p}_1),$$

$$E_{ij}^R \in \rho^\infty \Psi^{-\infty}$$

and Q^R having a similar decomposition to Q , but with off-diagonal terms in $\rho^\infty \Psi^{-2}$ and $\tilde{Q}_0^R \in \Psi_b^{-2, (\hat{S}, S \sqcup (\hat{S}+1), 1 \sqcup (S+\hat{S}) \cup 0)}(X; \mathfrak{p}_0)$.

A similar procedure gives a left parametrix Q^L with $E^L = I - Q^L F_X$ of the form

$$E^L = \begin{pmatrix} \rho^{1/2} \tilde{E}_0^L \rho^{-1/2} & E_{01}^L \\ E_{10}^L & E_1^L \end{pmatrix}$$

$$\tilde{E}_0^L \in \Psi_b^{-\infty, (\hat{S}, \infty, \infty)}(X; \mathfrak{p}_0), \quad E_1^L \in \Psi_{sc}^{-\infty, \infty}(X; \mathfrak{p}_1),$$

$$E_{ij}^L \in \rho^\infty \Psi^{-\infty}$$

As operators on $\rho^{\alpha+1, \beta} H_{b/sc}^{k, 0}(X; \mathfrak{p})$ and $\rho^{\alpha-1, \beta} H_{b/sc}^{k, 2}(X; \mathfrak{p})$, E^R and E^L are compact, so the extension (C.31) is Fredholm. From the fact that E^L maps $\rho^{\alpha-1, \beta} H_{b/sc}^{k, 2}$ into $\rho^{0, \infty} \mathcal{A}^{\hat{S}+1/2}$ and $u \in \text{Null}(F_X) \iff u = E^L u$, it follows that

$$\text{Null}(F_X) \subset \rho^{0, \infty} \mathcal{A}^{\hat{S}+1/2}(X; \mathfrak{p}).$$

In particular $u \in \text{Null}(F_X)$ has leading order $\mathcal{O}(\rho^{1+\epsilon})$ since $S \geq \frac{1}{2}$. This is enough decay to justify the integration by parts in the identity $\langle \Delta_A u, u \rangle_{L^2} = \|d_A u\|_{L^2}^2$, from which it follows that

$$u \in \text{Null}(F_X) \implies \|d_A u\|_{L^2} = 0,$$

so that u is covariant constant, and therefore vanishing since $u|_{\partial X} = 0$. Thus (C.31) is injective, and applying the same reasoning to the L^2 adjoint

$$\Delta_A - \text{ad } \Phi^2 : \rho^{-\alpha-1, -\beta} H_{\text{b/sc}}^{k,2}(X; \mathfrak{p}) \longrightarrow \rho^{-\alpha+1, -\beta} H_{\text{b/sc}}^{k,0}(X; \mathfrak{p})$$

proves that (C.31) is surjective as well. Letting G denote the inverse of (C.31) and writing

$$GF_X Q^R = G - GE^R = Q^R, \quad Q^L F_X G = G - E^L G = Q^L,$$

it follows that G satisfies the identity

$$G = Q^R + Q^L E^R + E^L G E^R.$$

(C.32) is then a consequence of the bi-ideal properties of $\rho^\infty \Psi^{-\infty}$ and the left (resp. right) ideal properties of $\Psi_{\text{b}}^{-\infty, (\infty, *, \infty)}$ (resp. $\Psi_{\text{b}}^{-\infty, (*, \infty, \infty)}$). \square

Next we consider the normal operators of F at \mathcal{D} . Again decomposing with respect to $\mathfrak{p}_0 \oplus \mathfrak{p}_1$, and writing

$$F = \begin{pmatrix} F_0 & F_{01} \\ F_{10} & F_1 \end{pmatrix},$$

with $F_{ij} = \mathcal{O}(\rho^\infty)$, we have

$$F_D := N_{\mathcal{D}}(F_0)|_D = (\lim_{\varepsilon \rightarrow 0} \varepsilon^2 F_0)|_D = \Delta_A|_D$$

where $A|_D$ is the restriction of A as a smooth connection to D .

Theorem C.12. *For $\alpha, \beta \in (-\frac{1}{2}, \frac{1}{2})$, and $k \in \mathbb{N}$, the extension*

$$\Delta_{A|D} : \rho_B^{\alpha-1} \rho_B^{\beta+1} H_{\text{b}}^{k+2}(D; \mathfrak{p}_0) \longrightarrow \rho_B^{\alpha+1} \rho_B^{\beta-1} H_{\text{b}}^k(D; \mathfrak{p}_0) \quad (\text{C.33})$$

is invertible, with inverse of the form

$$\begin{aligned} \Delta_{A|D}^{-1} &= \rho_B^{1/2} \rho_X^{-1/2} \tilde{G} \rho_X^{5/2} \rho_B^{-5/2}, \quad \tilde{G} \in \Psi_{\text{b}}^{-2, \mathcal{F}}(D; \mathfrak{p}_0), \\ \mathcal{F} &= (F_L, F_R, F_F), \quad F_L, F_R \geq \frac{1}{2}, \quad F_F \geq 0. \end{aligned} \quad (\text{C.34})$$

Proof. For the remainder of the proof, we denote $A|_D$ simply by A . We consider the operator

$$\tilde{\Delta}_A = \rho_B^{-5/2} \rho_X^{5/2} \Delta_A \rho_X^{-1/2} \rho_B^{1/2} \in \text{Diff}_{\text{b}}^2(D; \mathfrak{p}_0).$$

Taking A to be in radial gauge near the ends of D , it follows that near $D \cap X$,

$$\tilde{\Delta}_A = -(r\partial_r)^2 + \frac{1}{4} + \Delta_{A, D \cap X},$$

with $\Delta_{A, D \cap X}$ denoting the Laplacian on $D \cap X = \mathbb{S}^2$ induced by A and the metric $\rho_X^{-2} \varepsilon^{-2} g = h_{\mathbb{S}^2}$, and likewise near $D \cap B$,

$$\tilde{\Delta}_A = -(x\partial_x)^2 + \frac{1}{4} + \Delta_{A, D \cap B}.$$

In either case, the indicial roots near an end of D have the form

$$\text{spec}_{\text{b}}(\tilde{\Delta}_A) = \left\{ \pm \sqrt{\frac{1}{4} + \nu} : \nu \in \text{spec}(\Delta_{A, \partial D}) \right\}$$

and in particular are always disjoint from the interval $(-\frac{1}{2}, \frac{1}{2})$.

Proceeding with the standard steps in the b-calculus, we may construct right and left parametrices

$$\begin{aligned}\tilde{Q}^R &\in \Psi_b^{-2, (\hat{S}, S\sqcup(\hat{S}+1), 1\sqcup(S+\hat{S})\cup 0)}(D; \mathfrak{p}_0), \\ \tilde{Q}^L &\in \Psi_b^{-2, (S\sqcup(\hat{S}+1), \hat{S}, 1\sqcup(S+\hat{S})\cup 0)}(D; \mathfrak{p}_0)\end{aligned}$$

for $\tilde{\Delta}_A$, where $S = \{(s, k) \in \text{Spec}_b(\tilde{\Delta}_A) : s > 1/2\}$ and $\hat{S} = \overline{\bigcup}_n (S + n)$ as before. The error terms $\tilde{E}^R = I - \tilde{\Delta}_A \tilde{Q}^R$ and $\tilde{E}^L = I - \tilde{Q}^L \tilde{\Delta}_A$ have the form

$$\tilde{E}^L \in \Psi_b^{-\infty, (\hat{S}, \infty, \infty)}(D; \mathfrak{p}_0), \quad \tilde{E}^R \in \Psi_b^{-\infty, (\infty, \hat{S}, \infty)}(D; \mathfrak{p}_0).$$

Then $Q^{L/R} := \rho_B^{1/2} \rho_X^{-1/2} \tilde{Q}^{L/R} \rho_X^{5/2} \rho_B^{-5/2}$ are left/right parametrices for Δ_A , and the error terms $E^L = I - Q^L \Delta_A$ and $E^R = I - \Delta_A Q^R$ extend to compact operators on the domain and range of (C.33), respectively, with E^L mapping the domain into $\rho_B^{1/2} \rho_X^{-1/2} \mathcal{A}^{\hat{S}}(D; \mathfrak{p}_0)$.

It follows that (C.33) is Fredholm, with nullspace in $\rho_B^{1/2} \rho_X^{-1/2} \mathcal{A}^{\hat{S}}$; in particular, the leading order of $u \in \text{Null}(\Delta_A)$ has order $\mathcal{O}(\rho_B^1, \rho_X^0)$. Due to the homogeneity in the conic/scattering volume form of D , this is enough decay to justify the integration by parts in the identity $\langle \Delta_A u, u \rangle_{L^2} = \|d_A u\|_{L^2}^2$, from which it follows that $u \in \text{Null}(\Delta_A)$ is covariant constant, hence vanishing since it vanishes at $D \cap B$. Thus (C.33) is injective, and by self-adjointness is also surjective, hence invertible. (C.34) follows by writing

$$\Delta_A^{-1} = Q^R + Q^L E^R + E^L \Delta_A^{-1} E^R. \quad \square$$

As for F_1 , we recall the normal operator homomorphism of Proposition 5.4. Regarding $\sigma_{\mathcal{D}}(F_1)$ as a smooth fiberwise polynomial on ${}^\gamma T\mathcal{D} \rightarrow \mathcal{D}$ with values in $\text{End}(\mathfrak{p}_1)$, the observation that Δ_A is a laplacian on \mathfrak{p}_1 lead to the following Theorem.

Theorem C.13. *The symbol $\sigma_{\mathcal{D}}(F_1) \in S^2({}^\gamma T\mathcal{D}; \mathfrak{p}_1)$ is given at $(x, \xi) \in {}^\gamma T\mathcal{D}$ by*

$$\sigma_{\mathcal{D}}(F_1)(x, \xi) = |\xi|^2 - \text{ad } \Phi_x^2,$$

and is invertible for all (x, ξ) .

APPENDIX D. PSEUDODIFFERENTIAL OPERATORS

Here we construct the calculi of pseudodifferential operators which ‘microlocalize’ the algebras $\text{Diff}_\rho^*(\mathcal{Z})$ and $\text{Diff}_\gamma^*(\mathcal{Z})$ of differential operators associated to the vector fields $\mathcal{V}_\rho(\mathcal{Z})$ and $\mathcal{V}_\gamma(\mathcal{Z})$, respectively. To keep the notational complexity at a minimum we consider operators on scalar functions only; the extension to operators acting between sections of vector bundles over \mathcal{Z} is a straightforward matter.

The kernels of these pseudodifferential operators are defined as distributions on appropriate geometric resolutions of the fiber product $\mathcal{Z} \times_\rho \mathcal{Z}$; these

“double spaces” are discussed in §D.1. The composition of two such operators is defined via the associated “triple spaces”—resolutions of the triple fiber product $\mathcal{Z} \times_{\varrho} \mathcal{Z} \times_{\varrho} \mathcal{Z}$ —which are discussed next in §D.2. The ϱ and γ pseudodifferential operators are defined and some of their essential properties are investigated in §D.3 and their mapping properties with respect to Sobolev spaces are proved in §D.4.

Note that, in our constructions involving monopoles, we only compose pseudodifferential operators with differential ones. Such compositions can be defined directly on the double spaces, avoiding the technical complexity of the triple spaces. However, the composition of pseudodifferential operators is used to establish their mapping properties with respect to L^2 via a standard argument due to Hörmander (see the proof of Lemma D.9); for this reason we have developed the general composition results in Theorems D.7 and D.8 below.

D.1. The double spaces. In the first place, \mathcal{Z}_{ϱ}^2 is meant to be a resolution of the fiber product $\mathcal{Z}^{[2]} := \mathcal{Z} \times_{\varrho} \mathcal{Z}$ of the single space with itself with respect to the b-fibration $\varrho : \mathcal{Z} \rightarrow \mathcal{I} \times [0, \infty)$. There are two ways to achieve this. The most direct is to use the theory developed in [KM15] and [Kot15b] regarding resolutions of fiber products. Alternatively, since many readers will not be familiar with this theory, \mathcal{Z}_{ϱ}^2 may be constructed directly via a sequence of blow-ups from the space $(X)^2 \times \mathcal{I} \times [0, \infty)$. We shall describe both approaches.

Let Y be a manifold with corners and denote by $\mathcal{F}(Y) = \bigsqcup_d \mathcal{F}_d(Y)$ the set of boundary faces of Y , where d is codimension. Recall from [KM15, Kot15b] that Y has an associated *monoidal complex* \mathcal{P}_Y , which is a collection of monoids $\sigma_G \cong \mathbb{N}^d$ for each boundary face $G \in \mathcal{F}_d(Y)$, with canonical injective maps $i_{FG} : \sigma_G \hookrightarrow \sigma_F$ whenever $F \subset G$ identifying σ_G with a face of σ_F . These monoids σ_G are all *smooth*, meaning that they are freely generated by independent elements in the associated vector space $\sigma_G \otimes_{\mathbb{N}} \mathbb{R} \cong \mathbb{R}^d$; indeed the generators for a given σ_G may be identified with the faces $i_{GH}(\sigma_H) \cong \mathbb{N}$ for the hypersurfaces H meeting G . For any b-map $f : Y \rightarrow Z$, there is an associated morphism $f_{\natural} : \mathcal{P}_Y \rightarrow \mathcal{P}_Z$, mapping each σ_G into $\sigma_{f_{\#}(G)}$ where $f_{\#}(G)$ is the boundary face of Z of maximal codimension into which G maps. Expressed in a basis with respect to the generators of σ_G and $\sigma_{f_{\#}(G)}$, these maps of monoids are represented by matrices with nonnegative integer entries given by the exponents $e(H, H') \in \mathbb{N}$, where

$$f^*(\rho_H) = a \prod_{H' \in \mathcal{F}_1(Y)} \rho_{H'}^{e(H, H')}, \quad H \in \mathcal{F}_1(Z), \quad 0 < a \in C^{\infty}.$$

In fact, $f_{\natural} : \sigma_G \rightarrow \sigma_{f_{\#}(G)}$ may be identified with the b-differential

$${}^b f_* : {}^b N G \rightarrow {}^b N f_{\#}(G).$$

In particular, f is b-normal if and only if no generator of any σ_G is mapped into the interior of any monoid σ_F of \mathcal{P}_Z .

If $f_1 : Y_1 \rightarrow Z$ and $f_2 : Y_2 \rightarrow Z$ are b-maps which are *b-transversal*, a condition which is automatically satisfied if at least one is a b-fibration, then the fiber product

$$Y_1 \times_Z Y_2 \subset Y_1 \times Y_2 \quad (\text{D.1})$$

is what is called an *interior binomial variety* of the manifold $Y_1 \times Y_2$. Such a space, though smooth in its interior, is generally not a manifold with corners; it belongs to a category of “manifolds with generalized corners,” which has been developed by Joyce [Joy15], though subspaces such as (D.1) and their resolution theory is described in [KM15] and [Kot15b]. Indeed, a complex $\mathcal{P}_{Y_1 \times_Z Y_2}$ may also be associated to the fiber product, with each monoid of dimension l corresponding to a boundary face (which is again an interior binomial variety) of codimension l , and the face relations of the complex corresponding to the meeting of boundary faces. The failure of the fiber product to be smooth is measured by the failure of the monoids of $\mathcal{P}_{Y_1 \times_Z Y_2}$, which have the form $\sigma_{G_1} \times_{\sigma_F} \sigma_{G_2}$, $F = (f_1)_\#(G_1) \cap (f_2)_\#(G_2)$, to be smooth (meaning freely generated). To any *resolution* of $\mathcal{P}_{Y_1 \times_Z Y_2}$, meaning a consistent way of subdividing the non-smooth monoids into smooth ones, there corresponds an *generalized blow-up* of $Y_1 \times_Z Y_2$ resolving it to a smooth manifold with corners, the combinatorial structure of whose boundary faces (i.e., codimension and meeting of boundary faces) is again encoded by the resolving monoidal complex. Below we will also be concerned with the problem of determining the induced resolution of a boundary hypersurface $H \subset Y_1 \times_Z Y_2$ from a resolution of $\mathcal{P}_{Y_1 \times_Z Y_2}$. The monoidal complex of H itself is obtained from $\mathcal{P}_{Y_1 \times_Z Y_2}$ as the complex of quotient monoids $\{\sigma/\sigma_H : \sigma_H \subset \sigma \in \mathcal{P}_{Y_1 \times_Z Y_2}\}$, which we may denote by $\mathcal{P}_{Y_1 \times_Z Y_2}/\sigma_H$. A smooth resolution of $\mathcal{P}_{Y_1 \times_Z Y_2}$ induces a smooth resolution of the quotient since the quotient of a freely generated monoid by a generator is again freely generated.

In the present case of the b-fibration $\varrho : \mathcal{Z} \rightarrow \mathcal{I} \times [0, \infty)$, the space \mathcal{Z} has boundary faces of codimension at most 2, so $\mathcal{P}_{\mathcal{Z}}$ consists of the 1-dimensional monoids $\sigma_{\mathcal{X}_i}, \sigma_{\mathcal{D}}, \sigma_{\mathcal{B}} \cong \mathbb{N}$ and the 2-dimensional monoids $\sigma_{\mathcal{D} \cap \mathcal{B}}, \sigma_{\mathcal{D} \cap \mathcal{X}_i}$. The monoidal complex of the base $\mathcal{I} \times [0, \infty)$ consists of the single nontrivial monoid $\tau \cong \mathbb{N}$. The associated morphism ϱ_\sharp sends $\sigma_{\mathcal{B}}$ to $\{0\}$ while $\sigma_{\mathcal{D}}$ and $\sigma_{\mathcal{X}_i}$ are mapped isomorphically onto τ . On the monoids of dimension 2 we have

$$\begin{aligned} \varrho_\sharp : \sigma_{\mathcal{D} \cap \mathcal{X}_i} &\cong \mathbb{N}^2 \rightarrow \tau \cong \mathbb{N}, & (m, n) &\mapsto m + n, \\ \varrho_\sharp : \sigma_{\mathcal{D} \cap \mathcal{B}} &\cong \mathbb{N}^2 \rightarrow \tau \cong \mathbb{N}, & (m, n) &\mapsto m. \end{aligned}$$

It follows that the only singular monoids in $\mathcal{P}_{\mathcal{Z} \times_{\mathcal{I} \times [0, \infty)} \mathcal{Z}}$ are

$$\sigma_{\mathcal{D} \cap \mathcal{X}_i} \times_\tau \sigma_{\mathcal{D} \cap \mathcal{X}_j}, \quad 0 \leq i, j \leq N.$$

Each of these is a 3-dimensional monoid of the form

$$\nu := \{(m_1, n_1, m_2, n_2) \in \mathbb{N}^4 : m_1 + n_1 = m_2 + n_2\}, \quad (\text{D.2})$$

with dependent generators $(1, 0, 1, 0)$, $(0, 1, 0, 1)$, $(1, 0, 0, 1)$ and $(0, 1, 1, 0)$. These generators may be identified respectively with the four hypersurfaces

$$\mathcal{D} \times_{\mathcal{I}} \mathcal{D}, \mathcal{X}_i \times_{\mathcal{I}} \mathcal{X}_j, \mathcal{D} \times_{\mathcal{I}} \mathcal{X}_j, \mathcal{X}_i \times_{\mathcal{I}} \mathcal{D} \in \mathcal{F}_1(\mathcal{Z}^{[2]}) \quad (\text{D.3})$$

of the fiber product, which all meet at a face of codimension 3 (hence the singularity). Indeed, since \mathcal{D} and \mathcal{X} are both mapped to $\mathcal{U} \times \{0\} \subset \mathcal{I} \times [0, \infty)$, it follows that $\mathcal{D} \times_{\mathcal{I} \times [0, \infty)} \mathcal{D} \equiv \mathcal{D} \times_{\mathcal{I}} \mathcal{D}$, which is a fibration over \mathcal{I} with fibers D^2 . Likewise $\mathcal{D} \times_{\mathcal{I} \times [0, \infty)} \mathcal{X}_i \equiv \mathcal{D} \times_{\mathcal{I}} \mathcal{X}_i$ (with fibers $D \times X_i$ over \mathcal{I}) and so on, and these form boundary hypersurfaces of $\mathcal{Z}^{[2]}$ with associated monoids $\sigma_{\mathcal{D}} \times_{\tau} \sigma_{\mathcal{D}} \cong \mathbb{N}$, $\sigma_{\mathcal{D}} \times_{\tau} \sigma_{\mathcal{X}_i} \cong \mathbb{N}$, etc. The latter are identified with the generating 1-dimensional submonoids of $\sigma_{\mathcal{D} \cap \mathcal{X}_i} \times_{\tau} \sigma_{\mathcal{D} \cap \mathcal{X}_j}$ in the complex. For brevity of notation, we will denote the boundary hypersurfaces (D.3) as $\{\mathcal{D}, \mathcal{X}_{ij}, \mathcal{D}\mathcal{X}_j, \mathcal{D}\mathcal{X}_i\}$.

The task is to resolve the singular monoids above by smooth ones in a way which preserves the b-fibrations to the single space \mathcal{Z} ; these b-fibrations are represented by the monoid homomorphisms

$$\nu \longrightarrow \mathbb{N}^2, \quad (n_1, m_1, n_2, m_2) \longmapsto (n_i, m_i), \quad i = 1, 2. \quad (\text{D.4})$$

There are two inequivalent minimal resolutions, though in this case there is a canonical choice which to which the fiber diagonal lifts to be transversal to the boundary. This is to replace each monoid $\sigma_{\mathcal{D} \cap \mathcal{X}_i} \times_{\tau} \sigma_{\mathcal{D} \cap \mathcal{X}_j}$ by the two smooth monoids

$$\mathbb{N} \langle \mathcal{D}, \mathcal{D}\mathcal{X}_j, \mathcal{X}_{ij} \rangle, \quad \mathbb{N} \langle \mathcal{D}, \mathcal{X}_{ij}, \mathcal{D}\mathcal{X}_i \rangle, \quad (\text{D.5})$$

which is to say

$$\mathbb{N} \langle (1, 0, 1, 0), (1, 0, 0, 1), (0, 1, 0, 1) \rangle, \quad \mathbb{N} \langle (1, 0, 1, 0), (0, 1, 0, 1), (0, 1, 1, 0) \rangle,$$

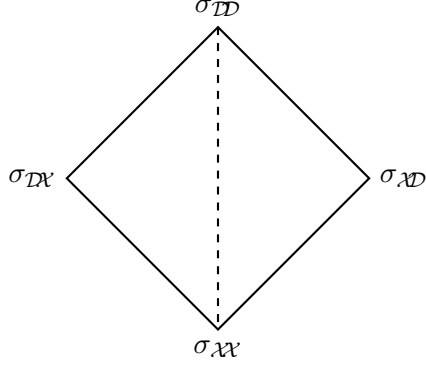
respectively. In doing so we replace each singular codimension 3 face by two smooth codimension 3 faces which are joined by a new codimension two face (corresponding to the monoid generated by $\{\sigma_{\mathcal{D}}, \sigma_{\mathcal{X}_{ij}}\}$). (See Figure 2.)

The result is a smooth manifold with corners we shall provisionally call \mathcal{Z}_{b}^2 with two b-fibrations to \mathcal{Z} lifting the right and left projection maps from the fiber product, and a b-fibration to the parameter space $\mathcal{I} \times [0, \infty)$. This is not yet the space we want; the final step is to blow-up the codimension 2 face represented by $\mathcal{B} \times_{\mathcal{I} \times [0, \infty)} \mathcal{B}$:

$$\mathcal{Z}_{\text{b}}^2 := [\mathcal{Z}_{\text{b}}^2; \mathcal{B} \times_{\mathcal{I} \times [0, \infty)} \mathcal{B}] \longrightarrow \mathcal{Z}_{\text{b}}^2. \quad (\text{D.6})$$

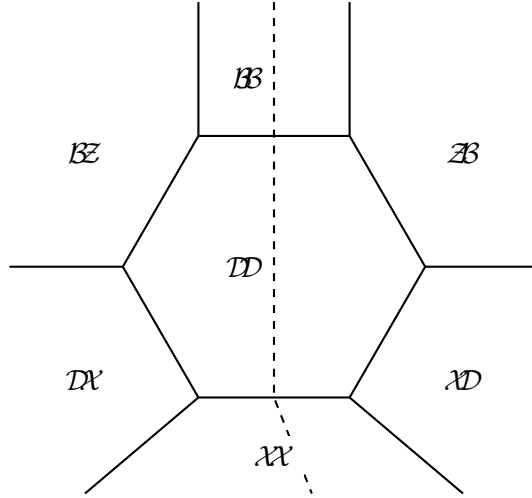
At the level of monoids, this corresponds to subdividing $\sigma_{\mathcal{B}^2} \cong \mathbb{N}^2$ into the two submonoids $\mathbb{N} \langle (1, 0), (1, 1) \rangle$ and $\mathbb{N} \langle (1, 1), (0, 1) \rangle$, with the new hypersurface represented by the common face $\mathbb{N} \langle (1, 1) \rangle$.

We denote the boundary hypersurfaces of \mathcal{Z}_{b}^2 by their factors in the original face of \mathcal{Z}_{b}^2 lifting the product $\mathcal{D} \times_{\mathcal{I} \times [0, \infty)} \mathcal{D} \equiv \mathcal{D} \times_{\mathcal{I}} \mathcal{D}$ in $\mathcal{Z}^{[2]}$; below we show that it is diffeomorphic to the families b double space $(\mathcal{D} \times_{\mathcal{I}} \mathcal{D})_{\text{b}}$.

FIGURE 2. Smooth refinement of the monoid $\sigma_{\mathcal{DK}} \times_{\tau} \sigma_{\mathcal{DK}}$

Similarly, \mathcal{DK}_i denotes the lift of $\mathcal{D} \times_{\mathcal{I} \times [0, \infty)} \mathcal{K}_i$ and so on. We write \mathcal{B} to denote the front face of the blow-up (D.6), and \mathcal{BZ} , etc. to denote the hypersurface lifting the original face $\mathcal{B} \times_{\mathcal{I} \times [0, \infty)} \mathcal{Z}$ of the fiber product. The complete list of boundary hypersurfaces of \mathcal{Z}_{ϱ}^2 is as follows (see Figure 3):

$$\mathcal{D}, \quad \mathcal{DK}_i, \quad \mathcal{KD}_i, \quad \mathcal{KK}_{ij}, \quad \mathcal{B}, \quad \mathcal{BZ}, \quad \mathcal{BZ},$$

FIGURE 3. The double space \mathcal{Z}_{ϱ}^2 .

Lemma D.1. (a) *The double space \mathcal{Z}_{ϱ}^2 has b-fibrations $\pi_R : \mathcal{Z}_{\varrho}^2 \rightarrow \mathcal{Z}$ and $\pi_L : \mathcal{Z}_{\varrho}^2 \rightarrow \mathcal{Z}$ and $\varrho : \mathcal{Z}_{\varrho}^2 \rightarrow \mathcal{I} \times [0, \infty)$ lifting the corresponding maps from the fiber product $\mathcal{Z}^{[2]} = \mathcal{Z} \times_{\varrho} \mathcal{Z}$. Composing with the projection*

- $\mathcal{I} \times [0, \infty) \longrightarrow \mathcal{I}$ defines a fiber bundle $\mu : \mathcal{Z}_\varrho^2 \longrightarrow \mathcal{I}$ whose fibers, denoted generically by Z_ϱ^2 , are manifolds with corners.
- (b) The lifted fiber diagonal $\Delta \subset \mathcal{Z}_\varrho^2$ meets all boundary faces transversally.
 - (c) The boundary faces meeting Δ may be identified with the family (over \mathcal{I}) b-double spaces $(\mathcal{D} \times_{\mathcal{I}} \mathcal{D})_{\text{b}}$, $(\mathcal{X}_i \times_{\mathcal{I}} \mathcal{X}_i)_{\text{b}}$, and the front face of the blow-up (D.6), which may be identified with the product of the b-front face of $(\mathbb{R}^3)_{\text{b}}^2$ with $\mathcal{I} \times [0, \infty)$.
 - (d) For $\varepsilon > 0$, the fibers $\varrho^{-1}(\{p\} \times \{\varepsilon\}) \subset \mathcal{Z}_\varrho^2$, $p \in \mathcal{I}$ are diffeomorphic to the b double space X_{b}^2 .

Remark. $(\mathcal{D} \times_{\mathcal{I}} \mathcal{D})_{\text{b}}$ is a fiber bundle over \mathcal{I} with fibers given by the b-double space D_{b}^2 , obtained by blowing up the coimension 2 corner in the fiber product $\mathcal{D} \times_{\mathcal{I}} \mathcal{D}$. It supports Schwartz kernels of families of b-pseudodifferential operators on the fiberr of $\mathcal{D} \longrightarrow \mathcal{I}$, which we denote by $\Psi_{\text{b}}^*(\mathcal{D}/\mathcal{I})$. Similar statements hold for $(\mathcal{X}_i \times_{\mathcal{I}} \mathcal{X}_i)_{\text{b}}$ and $(\mathcal{X}_\varepsilon \times_{\mathcal{I}} \mathcal{X}_\varepsilon)_{\text{b}}$.

Proof. In the first place, the space \mathcal{Z}_{b}^2 has b-fibrations to \mathcal{Z} and $\mathcal{I} \times [0, \infty)$, lifting the assocaited maps from the fiber product. Indeed, since only boundary faces have been blown up in passing from $\mathcal{Z}^{[2]}$ to \mathcal{Z}_{b}^2 , the only issue to check is b-normality, and this can be verified at the level of the monoids. The homomorphisms (D.4) from singular monoids $\sigma_{\mathcal{D} \cap \mathcal{X}_i} \times_{\tau} \sigma_{\mathcal{D} \cap \mathcal{X}_j}$ have the b-normality property, which is that no 1-dimensional face of a monoid is mapped into the interior of a monoid in the range complex $\mathcal{P}_{\mathcal{Z}}$, and the resolution described above retains this property. The subsequent blow-up of $\mathcal{B} \times_{\mathcal{I} \times [0, \infty)} \mathcal{B}$ to define \mathcal{Z}_ϱ^2 introduces a new boundary hypersurface, but this is mapped via the right and left projections to the hypersurface $\mathcal{B} \subset \mathcal{Z}$, so the lifted maps from \mathcal{Z}_ϱ^2 are b-fibrations as well. Since b-fibrations are closed under composition and $\mathcal{I} \times [0, \infty) \longrightarrow \mathcal{I}$ is a b-fibration to a manifold with no boundary, (a) follows.

For (b) and (c) consider the boundary faces of the singular space $\mathcal{Z}^{[2]}$ which meet Δ . These are evidently the faces $\mathcal{D}^{[2]} \equiv \mathcal{D} \times_{\mathcal{I}} \mathcal{D}$, $\mathcal{X}_i^{[2]} \equiv \mathcal{X}_i \times_{\mathcal{I}} \mathcal{X}_i$ (which are hypersurfaces as remarked above), $\mathcal{B}^{[2]}$ (which has codimension 2), the singular corners $(\mathcal{D} \cap \mathcal{X}_i)^{[2]}$ of codimension 3, and the smooth codimension 3 corner $(\mathcal{D} \cap \mathcal{B})^{[2]}$. Upon resolution, $\mathcal{B}^{[2]}$ is replaced by a hypersurface, with the lifted fiber diagonal meeting it transversally, and the codimension 3 faces above are replaced by codimension 2 corners.

By considering the quotient monoids $\sigma_{\mathcal{D} \cap \mathcal{X}_i} \times_{\tau} \sigma_{\mathcal{D} \cap \mathcal{X}_j} / \sigma_{\mathcal{D}} \cong \mathbb{N}^2$ and $\sigma_{\mathcal{D} \cap \mathcal{B}} \times_{\tau} \sigma_{\mathcal{D} \cap \mathcal{B}} / \sigma_{\mathcal{D}} \cong \mathbb{N}^2$, and the corresponding quotients of the resolution, which induce the ordinary blow-up (i.e., star subdivision) of \mathbb{N}^2 , it follows that the lift of $\mathcal{D}^{[2]}$ to \mathcal{Z}_ϱ^2 corresponds to the the blow-up $[\mathcal{D}^{[2]}; (\mathcal{D} \cap \mathcal{B})^{[2]}, (\mathcal{D} \cap \mathcal{X}_i)^{[2]}]$, which is precisely the b-double space $(\mathcal{D} \times_{\mathcal{I}} \mathcal{D})_{\text{b}}$. A similar argument shows that the hypersurfaces $\mathcal{X}_i^{[2]} \subset \mathcal{M}^{[2]}$ lift to the b-double spaces $(\mathcal{X}_i \times_{\mathcal{I}} \mathcal{X}_i)_{\text{b}}$. The lifted fiber diagonal passes through the front faces of $(\mathcal{D} \times_{\mathcal{I}} \mathcal{D})_{\text{b}}$ and $(\mathcal{X}_i \times_{\mathcal{I}} \mathcal{X}_i)_{\text{b}}$, and is therefore a p-submanifold. This

can be verified directly in the monoidal theory; the monoidal subcomplex $\mathcal{P}_\Delta \subset \mathcal{P}_{\mathcal{Z} \times_{\mathcal{I} \times [0, \infty)} \mathcal{Z}}$ meets each singular monoid of the form ν in the submonoid generated by $\{(1, 0, 1, 0), (0, 1, 0, 1)\}$ and meets $\sigma_{\mathcal{B} \times_{\mathcal{I} \times [0, \infty)} \mathcal{B}} \cong \mathbb{N}^2$ in the submonoid generated by $(1, 1)$. In the above resolution, we introduced each of these into the complex, and by a result in [KM15], it follows that Δ lifts to a p-submanifold in the resolution.

Finally, to see (d), note that the fiber of $\mathcal{Z}^{[2]}$ over $\varepsilon > 0$ is smooth, and coincides with the product $\varrho^{-1}(\varepsilon) \times_{\mathcal{U}} \varrho^{-1}(\varepsilon) \cong X^2 \times \mathcal{I}$. This is unchanged in passing to \mathcal{Z}_b^2 , since all blow-ups take place at $\varepsilon = 0$. The blow-up of $\mathcal{B} \times_{\mathcal{I} \times [0, \infty)} \mathcal{B}$ then restricts to the blow up $[X^2 \times \mathcal{I}; (\partial X)^2 \times \mathcal{I}]$ in the fiber over ε , giving the b-double space as claimed. \square

Alternatively, \mathcal{Z}_ϱ^2 may be constructed as follows. For simplicity we restrict consideration to a single fiber Z of $\varrho : \mathcal{Z} \rightarrow \mathcal{I} \times [0, \infty)$. First, note that Z itself may be obtained as a blow up of the product $D \times [0, 1]$, where $D = \overline{\mathbb{R}^3}$:

$$Z = [D \times [0, 1]; \{p_0, p_1, \dots, p_N\} \times \{0\}],$$

where the collection of points $\{p_1, \dots, p_N\}$ at $\varepsilon = 0$ is determined by the appropriate configuration in \mathcal{E}_N^* , and $p_0 = 0$ is the origin in D . That this is equivalent to the definition of \mathcal{Z} in §3 can be checked in local coordinates. The intermediate space \mathcal{Z}_b^2 is obtained by iterated blow-up; fiberwise

$$\mathcal{Z}_b^2 = [D^2 \times [0, 1]; \{p_i \times p_j\} \times \{0\}, \{p_i \times D\} \times \{0\}, \{D \times p_j\} \times \{0\}].$$

First the pairs $p_i \times p_j$ are blown up at $\varepsilon = 0$ (they are separated so the order is not important); once this is done the lifts of the subspaces $p_i \times D \times \{0\}$ and $D \times p_j \times \{0\}$ are separated and may be blown up in any order. The final step is the blow-up (D.6). That the resulting space satisfies the properties in Lemma D.1 may be verified by straightforward but tedious computations in local coordinates.

Since the interiors of \mathcal{Z} and \mathcal{Z}_ϱ^2 may be identified with the simple products $\mathbb{R}^3 \times \mathcal{I} \times (0, \infty)$ and $(\mathbb{R}^3)^2 \times \mathcal{I} \times (0, \infty)$, respectively, the lift of vector fields in $\mathcal{V}_\varrho(\mathcal{Z})$ to \mathcal{Z}_ϱ^2 from the left or right is well-defined by continuous extension from the interior, and we have the following result:

Lemma D.2. *Let $V \in \mathcal{V}_\varrho(\mathcal{Z})$. The left and right lifts $\pi_L^*(V)$ and $\pi_R^*(V)$ are tangent to all boundary faces of \mathcal{Z}_ϱ^2 , and differentiate transversally to Δ . In particular, the restriction of the lift $\pi_R^*(V)$ to the boundary faces \mathcal{D} , \mathcal{X}_{ii} , or to the fiber $\mathcal{X}_\varepsilon^2 := \pi_{[0,1]}^{-1}(\varepsilon)$ for $\varepsilon > 0$ may be respectively identified with the following:*

$$\pi_R^*(V)|_{\mathcal{D}} \cong \pi_R^*(V|_{\mathcal{D}}), \quad \pi_R : (\mathcal{D} \times_{\mathcal{I}} \mathcal{D})_b \rightarrow \mathcal{D} \subset \mathcal{Z}, \quad (\text{D.7a})$$

$$\pi_R^*(V)|_{\mathcal{X}_{ii}} \cong \pi_R^*(V|_{\mathcal{X}_i}), \quad \pi_R : (\mathcal{X}_i \times_{\mathcal{I}} \mathcal{X}_i)_b \rightarrow \mathcal{X}_i \subset \mathcal{Z}, \quad (\text{D.7b})$$

$$\pi_R^*(V)|_{\mathcal{X}_\varepsilon^2} \cong \pi_R^*(V|_{\mathcal{X}_\varepsilon}), \quad \pi_R : (\mathcal{X}_\varepsilon \times_{\mathcal{I}} \mathcal{X}_\varepsilon)_b \rightarrow \mathcal{X}_\varepsilon \subset \mathcal{Z}, \quad (\text{D.7c})$$

and similarly for the restriction of $\pi_L^*(V)$.

Proof. Restricted to the interior, the left and right lifts of $\mathcal{V}_\varrho(\mathcal{Z})$ to \mathcal{Z}_ϱ^2 are readily seen to differentiate transversally to Δ , so it remains to determine their behavior at the boundary faces.

The restriction of $\pi_R : \mathcal{Z}_\varrho^2 \rightarrow \mathcal{Z}$ to $\mathcal{TD} \cong (\mathcal{D} \times_{\mathcal{I}} \mathcal{D})_{\text{b}}$ factors through the corresponding face of the fiber product $\mathcal{Z}^{[2]}$, which as noted above is simply the product $\mathcal{D} \times_{\mathcal{I}} \mathcal{D}$, followed by projection onto the right factor, realized as the boundary face of \mathcal{Z} . It follows that

$$\pi_R|_{\mathcal{TD}} : \mathcal{TD} \cong (\mathcal{D} \times_{\mathcal{I}} \mathcal{D})_{\text{b}} \rightarrow \mathcal{D}$$

is identified with the conventional right projection from the b-double space. The corresponding statements for the left projection, and the restriction of π_R and π_L to the other boundary faces and the fiber $(\mathcal{X}_\varepsilon \times_{\mathcal{I}} \mathcal{X}_\varepsilon)_{\text{b}}$ are similar.

The identifications (D.7a)–(D.7c) follow immediately, and the result that $\pi_R^*(V)$ and $\pi_L^*(V)$ are tangent to the boundary faces away from Δ follows from factoring the restriction of π_R or π_L through similar product faces of $\mathcal{Z}^{[2]}$. \square

The lift of vector fields from the left or right extends to differential operators. Thus for $D \in \text{Diff}_\varrho^*(\mathcal{Z}; V)$, the pull-backs $\pi_L^*(D)$ and $\pi_R^*(D)$ are well-defined.

Next we consider the gluing double space \mathcal{Z}_γ^2 . This is obtained from \mathcal{Z}_ϱ^2 by a sequence of two blow-ups:

$$\mathcal{Z}_\gamma^2 = [\mathcal{Z}_\varrho^2; \Delta \cap \mathcal{B}\mathcal{Z}; \Delta \cap \mathcal{TD}].$$

We denote the new boundary hypersurfaces by \mathcal{B}_{sc} and \mathcal{D}_{sc} , respectively. (See Figure 4.)

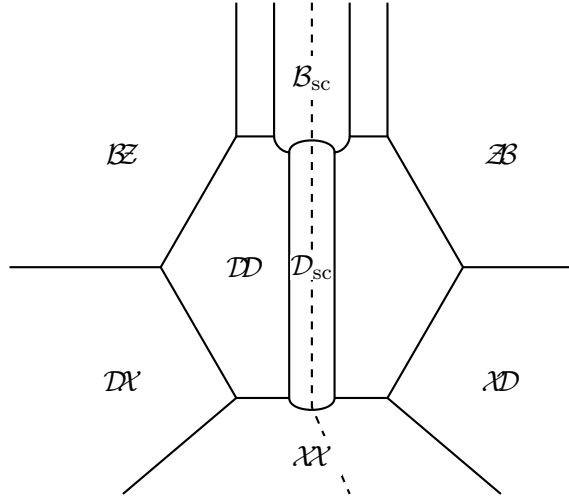


FIGURE 4. The double space \mathcal{Z}_γ^2 .

Lemma D.3. *The b-fibrations $\pi_R, \pi_L : \mathcal{Z}_\rho^2 \rightarrow \mathcal{Z}$ lift to b-fibrations of \mathcal{Z}_γ^2 to \mathcal{Z} . The boundary hypersurface \mathcal{D}_{sc} is diffeomorphic to the fiberwise radial compactification $\overline{\gamma T\mathcal{D}} \rightarrow \mathcal{D}$. The lift of \mathcal{X}_{ii} to \mathcal{Z}_γ^2 is diffeomorphic to the scattering double space $(\mathcal{X}_i \times_{\mathcal{I}} \mathcal{X}_i)_{\text{sc}}$ (with fibers $(X_i)_{\text{sc}}^2$ over \mathcal{I}), as is the lift of any $\varepsilon > 0$ fiber.*

The scattering face of $(\mathcal{X}_i \times_{\mathcal{I}} \mathcal{X}_i)_{\text{sc}}$ is identified with the fiber $\overline{\gamma T\mathcal{D}}|_{\mathcal{D} \cap \mathcal{X}_i}$.

Proof. To show that π_R and π_L lift to b-fibrations it suffices to show that no boundary hypersurface of \mathcal{Z}_γ^2 is mapped into a boundary face of \mathcal{Z} of codimension more than 1 under π_R or π_L . For the lifts of boundary hypersurfaces from \mathcal{Z}_ρ^2 , this follows from the fact that the original maps on \mathcal{Z}_ρ^2 are b-fibrations. This leaves only \mathcal{D}_{sc} and \mathcal{B}_{sc} , which are mapped into the hypersurfaces \mathcal{D} and \mathcal{B} , respectively.

As noted previously, $\mathcal{X}_{ii} \subset \mathcal{Z}_\rho^2$ is diffeomorphic to the b-double space $(\mathcal{X}_i \times_{\mathcal{I}} \mathcal{X}_i)_{\text{b}}$. The only blow-up which affects this boundary face is the blow up of $\Delta \cap \mathcal{D}$, which meets $(\mathcal{X}_i \times_{\mathcal{I}} \mathcal{X}_i)_{\text{b}}$ at the intersection $\Delta \cap \text{bf} \subset (\mathcal{X}_i \times_{\mathcal{I}} \mathcal{X}_i)_{\text{b}}$ of the diagonal with the b-front face. It follows that that lift of \mathcal{X}_{ii} to \mathcal{Z}_γ^2 is diffeomorphic to $(\mathcal{X}_i \times_{\mathcal{I}} \mathcal{X}_i)_{\text{sc}}$. A similar argument applies to $\mathcal{X}_\varepsilon^2$ which likewise meets the blow-up locus $\Delta \cap \mathcal{B}$ at $\Delta \cap \text{bf}$ under the identification $\mathcal{X}_\varepsilon^2 \cong (\mathcal{X}_\varepsilon \times_{\mathcal{I}} \mathcal{X}_\varepsilon)_{\text{b}}$.

For the identification of \mathcal{D}_{sc} , we work with local coordinates. Local coordinates on \mathcal{Z}_ρ^2 near $\mathcal{D} \cap \mathcal{X}_{ii}$ are furnished by (x, r', s, y, y', q) , where $s = \frac{x'}{x} = \frac{r}{r'}$ and x is boundary defining for \mathcal{D} . Passing to the blow-up, these give local coordinates

$$(x, r', \sigma, \eta, y', q), \quad \sigma = \frac{s-1}{x} \in \mathbb{R}, \quad \eta = \frac{y'-y}{x} \in \mathbb{R}^2,$$

on \mathcal{D}_{sc} (with boundary defining coordinate x) near \mathcal{X}_{ii} but away from its intersection with \mathcal{D} . This front face has the structure of a (radially compactified) vector bundle over \mathcal{D} where $(\sigma, \eta) \in \mathbb{R}^3$ are the fiber coordinates and the projection is given by $(0, r', \sigma, \eta, y', q) \mapsto (r', y', q)$. To identify this with $\gamma T\mathcal{D}$, we consider the generating vector fields $\{\frac{1}{2}(xr\partial_r - x^2\partial_x), x\partial_y\}$, which lift to

$$\{\partial_\sigma, \partial_\eta\} + \mathcal{O}(x)$$

Thus the coordinate basis $\{\partial_\sigma, \partial_\eta\}$ for the vector space constituting a fiber of \mathcal{D}_{sc} over is naturally associated to the basis of a fiber of $\gamma T\mathcal{D}$. For later use, we record the fact that the lift of $\{\frac{1}{2}(x'r'\partial_{r'} - (x')^2\partial_{x'}, x'\partial_{y'}\}$ (i.e., the lift of gluing vector fields from the left) is given by $\{-\partial_\sigma, -\partial_\eta\} + \mathcal{O}(x)$.

At the other end, where \mathcal{D}_{sc} meets \mathcal{B}_{sc} , we start with the local coordinates $(x, \varepsilon, s, y, y', q)$ on \mathcal{Z}_ρ^2 near $\mathcal{D} \cap \mathcal{B}$, where here $s = \frac{x'}{x}$ and now x and x' are the right and left lifts of the boundary defining function $\rho_{\mathcal{B}}$ for the big end $\mathcal{D} \cap \mathcal{B}$ of \mathcal{D} . The first blow-up, of $\Delta \cap \mathcal{B}$ is represented by coordinates

$$(x, \varepsilon, \sigma, \eta, y', q), \quad \sigma = \frac{s-1}{x}, \quad \eta = \frac{y'-y}{x},$$

where now (σ, η) represent coordinates on the scattering tangent bundle ${}^{\text{sc}}T\mathcal{D}$. The subsequent blow-up of $\Delta \cap \mathcal{D}$ (given locally by $\{\varepsilon = \sigma = \eta = 0\}$)

is then represented by

$$(x, \varepsilon, \varsigma, \xi, y', q), \quad \varsigma = \frac{\sigma}{\varepsilon}, \quad \xi = \frac{\eta}{\varepsilon},$$

where here $(\varsigma, \xi) \in \mathbb{R}^3$ are fiber coordinates on $\mathcal{D}_{\text{sc}} \rightarrow \mathcal{D}$ and ε is boundary defining for \mathcal{D}_{sc} . The generating vector fields $\{\varepsilon x^2 \partial_x, \varepsilon x \partial_y\}$ lift to

$$\{\partial_\varsigma, \partial_\xi\} + \mathcal{O}(\varepsilon)$$

from the right, and to $\{-\partial_\varsigma, -\partial_\xi\} + \mathcal{O}(x)$ from the left. \square

As with the ϱ double space, we may lift vector fields in $\mathcal{V}_\gamma(\mathcal{Z})$ to \mathcal{Z}_γ^2 from the left or right; these lifts are well-defined by continuity from the interior using the product decomposition $\mathcal{Z}_\gamma^2 \cong (\mathbb{R}^3)^2 \times \mathcal{I} \times (0, \infty)$, and we have

Lemma D.4. *The left/right lifts of $V \in \mathcal{V}_\gamma(\mathcal{Z})$ are tangent to all boundary faces of \mathcal{Z}_γ^2 and are differentially transversal to Δ . The restriction of $\pi_R^*(V)$ to \mathcal{D}_{sc} , \mathcal{X}_{ii} or $\mathcal{X}_\varepsilon^2$ may be identified with the lift of the restriction to the corresponding double space:*

$$\pi_R^*(V)|_{\mathcal{D}_{\text{sc}}} \cong \pi_R^*(V|_{\mathcal{D}}), \quad \pi_R : \overline{\gamma T \mathcal{D}} \rightarrow \mathcal{D} \subset \mathcal{Z}, \quad (\text{D.8a})$$

$$\pi_R^*(V)|_{\mathcal{X}_{ii}} \cong \pi_R^*(V|_{\mathcal{X}_i}), \quad \pi_R : (\mathcal{X}_i \times_{\mathcal{I}} \mathcal{X}_i)_{\text{sc}} \rightarrow \mathcal{X}_i \subset \mathcal{Z}, \quad (\text{D.8b})$$

$$\pi_R^*(V)|_{\mathcal{X}_\varepsilon^2} \cong \pi_R^*(V|_{\mathcal{X}_\varepsilon}), \quad \pi_R : (\mathcal{X}_\varepsilon \times_{\mathcal{I}} \mathcal{X}_\varepsilon)_{\text{sc}} \rightarrow \mathcal{X}_\varepsilon \subset \mathcal{Z}, \quad (\text{D.8c})$$

and similarly for $\pi_L^*(V)$.

Proof. For \mathcal{X}_{ii} and $\mathcal{X}_\varepsilon^2$, the result follows from Lemma D.2, and Lemma D.3. The result for \mathcal{D}_{sc} follows from the computations done in the proof of Lemma D.3. \square

D.2. The triple spaces. To obtain the triple space \mathcal{Z}_ϱ^3 , we start with the binomial subvariety (manifold with generalized corners)

$$\mathcal{Z}^{[3]} := \mathcal{Z} \times_\varrho \mathcal{Z} \times_\varrho \mathcal{Z} \subset \mathcal{Z}^3,$$

and its monoidal complex, consisting of monoids of the form $\sigma_\bullet \times_\tau \sigma'_\bullet \times_\tau \sigma''_\bullet$, where $\sigma_\bullet, \sigma'_\bullet, \sigma''_\bullet \in \{\sigma_{\mathcal{D}}, \sigma_{\mathcal{X}_i}, \sigma_{\mathcal{B}}\}$. Of these, the most singular are $\sigma_{\mathcal{D} \cap \mathcal{X}_i} \times_\tau \sigma_{\mathcal{D} \cap \mathcal{X}_j} \times_\tau \sigma_{\mathcal{D} \cap \mathcal{X}_k}$, which are each isomorphic to the 4-dimensional monoid

$$\mu := \{(n_1, m_1, n_2, m_2, n_3, m_3) : n_1 + m_1 = n_2 + m_2 = n_3 + m_3\} \subset \mathbb{N}^6 \quad (\text{D.9})$$

with the 8 generators

$$\begin{aligned} (1, 0, 1, 0, 1, 0), & \quad (1, 0, 1, 0, 0, 1), \\ (1, 0, 0, 1, 1, 0), & \quad (1, 0, 0, 1, 0, 1), \\ (0, 1, 1, 0, 1, 0), & \quad (0, 1, 1, 0, 0, 1), \\ (0, 1, 0, 1, 1, 0), & \quad (0, 1, 0, 1, 0, 1). \end{aligned}$$

These may be identified respectively with the hypersurfaces (introducing the obvious notation) $\mathcal{DD}, \mathcal{DX}_k, \mathcal{D}\mathcal{D}_j, \dots, \mathcal{XX}_{ijk}$. To resolve these, we subdivide each such monoid into the 6 smooth submonoids:

$$\begin{aligned} \mathbb{N} \langle \mathcal{DD}, \mathcal{DX}_k, \mathcal{DX}_{jk}, \mathcal{XX}_{ijk} \rangle, & \quad \mathbb{N} \langle \mathcal{DD}, \mathcal{DX}_k, \mathcal{DX}_{ik}, \mathcal{XX}_{ijk} \rangle, \\ \mathbb{N} \langle \mathcal{DD}, \mathcal{D}\mathcal{D}_j, \mathcal{DX}_{jk}, \mathcal{XX}_{ijk} \rangle, & \quad \mathbb{N} \langle \mathcal{DD}, \mathcal{D}\mathcal{D}_j, \mathcal{XD}_{ij}, \mathcal{XX}_{ijk} \rangle, \\ \mathbb{N} \langle \mathcal{DD}, \mathcal{XD}_i, \mathcal{DX}_{ik}, \mathcal{XX}_{ijk} \rangle, & \quad \mathbb{N} \langle \mathcal{DD}, \mathcal{XD}_i, \mathcal{XD}_{ij}, \mathcal{XX}_{ijk} \rangle. \end{aligned} \quad (\text{D.10})$$

The remaining singular monoids are of the form $\sigma_{\mathcal{D} \cap \mathcal{X}_i} \times_{\tau} \sigma_{\mathcal{D} \cap \mathcal{X}_j} \times_{\tau} \sigma_{\mathcal{B}}$ (and various permutations of the factors), which are products of the form $\nu \times \mathbb{N}$, where ν is the monoid in (D.2) and are resolved by taking the product of the resolution of ν discussed above with \mathbb{N} . The result of all this is a smooth manifold $\mathcal{Z}_{\mathfrak{b}}^3$, and the final step is the blow-up

$$\mathcal{Z}_{\mathfrak{c}}^3 := [\mathcal{Z}_{\mathfrak{b}}^3; \mathcal{BB}, \mathcal{BZ}, \mathcal{BZ}, \mathcal{AB}].$$

This corresponds to the subdivision of $\sigma_{\mathcal{BB}} \cong \mathbb{N}^3$ (generated by the 1-dimensional submonoids $\sigma_{\mathcal{BZ}} = \mathbb{N} \langle (1, 0, 0) \rangle$, $\sigma_{\mathcal{AZ}} = \mathbb{N} \langle (0, 1, 0) \rangle$, and $\sigma_{\mathcal{ZZ}} = \mathbb{N} \langle (0, 0, 1) \rangle$) into the following 6 submonoids:

$$\begin{aligned} \mathbb{N} \langle (1, 0, 0), (1, 0, 1), (1, 1, 1) \rangle, & \quad \mathbb{N} \langle (1, 0, 0), (1, 1, 0), (1, 1, 1) \rangle, \\ \mathbb{N} \langle (0, 1, 0), (0, 1, 1), (1, 1, 1) \rangle, & \quad \mathbb{N} \langle (0, 1, 0), (1, 1, 0), (1, 1, 1) \rangle, \\ \mathbb{N} \langle (0, 0, 1), (1, 0, 1), (1, 1, 1) \rangle, & \quad \mathbb{N} \langle (0, 0, 1), (0, 1, 1), (1, 1, 1) \rangle. \end{aligned} \quad (\text{D.11})$$

(This is equivalent to the so-called “total boundary blow-up” of the codimension 3 corner \mathcal{BB} .)

As with $\mathcal{Z}_{\mathfrak{c}}^2$, we label the boundary hypersurfaces of $\mathcal{Z}_{\mathfrak{c}}^3$ by the boundary faces of $\mathcal{Z}^{[3]}$ which they lift, which are labeled in turn by the corresponding products of boundary hypersurfaces of \mathcal{Z} , omitting the symbol $\times_{\mathcal{I}}$. The complete list of these is:

$$\begin{aligned} \mathcal{DD}, \mathcal{DX}_k, \mathcal{D}\mathcal{D}_j, \mathcal{XD}_i, \mathcal{DX}_{jk}, \mathcal{XD}_{ij}, \mathcal{DX}_{ik}, \mathcal{XX}_{ijk}, \\ \mathcal{BB}, \mathcal{BZ}, \mathcal{BZ}, \mathcal{AB}, \mathcal{BZ}, \mathcal{ZB}, \mathcal{AZ}. \end{aligned}$$

Lemma D.5.

- (a) $\mathcal{Z}_{\mathfrak{c}}^3$ admits three b -fibrations $\mathcal{Z}_{\mathfrak{c}}^3 \rightarrow \mathcal{Z}_{\mathfrak{c}}^2$ lifting the 3 projections $\mathcal{Z}^{[3]} \rightarrow \mathcal{Z}^{[2]}$.
- (b) The total and partial fiber diagonals in $\mathcal{Z}^{[3]}$ lift to p -submanifolds of $\mathcal{Z}_{\mathfrak{c}}^3$.
- (c) The boundary hypersurface \mathcal{DD} is diffeomorphic to the b -triple space $(\mathcal{D} \times_{\mathcal{I}} \mathcal{D} \times_{\mathcal{I}} \mathcal{D})_{\mathfrak{b}}$ (with fiber $D_{\mathfrak{b}}^3$ over \mathcal{I}) and likewise $\mathcal{XX}_{iii} \cong (\mathcal{X} \times_{\mathcal{I}} \mathcal{X} \times_{\mathcal{I}} \mathcal{X})_{\mathfrak{b}}$ (with fiber $X_{\mathfrak{b}}^3$). The lifted projections restrict over these hypersurfaces to the corresponding lifted projections to the b -double spaces.
- (d) For $\varepsilon > 0$, the fiber $\mathcal{XX}_{\varepsilon} := \varrho^{-1}(\mathcal{I} \times \{\varepsilon\})$ of $\varrho : \mathcal{Z}_{\mathfrak{c}}^3 \rightarrow \mathcal{I} \times [0, \infty)$ is diffeomorphic to the b triple space $(\overline{\mathbb{R}^3})_{\mathfrak{b}}^3 \times \mathcal{I}$.

Proof. In the first place, observe that, under the 3 projections $\mathbb{N}^6 = (\mathbb{N}^2)^3 \rightarrow \mathbb{N}^4 = (\mathbb{N}^2)^2$, the resolution (D.10) projects to the resolution (D.5) of (D.2) described above. By results in [KM15, Kot15b], it follows that the maps

$\mathcal{Z}^{[3]} \longrightarrow \mathcal{Z}^{[2]}$ lift to well-defined b-maps $\mathcal{Z}_\rho^3 \longrightarrow \mathcal{Z}_\rho^2$. That they are b-fibrations follows from the fact that, under the corresponding morphisms of monoids, no 1-dimensional monoid of $\mathcal{P}_{\mathcal{Z}_\rho^3}$ is mapped into the interior of a monoid in $\mathcal{P}_{\mathcal{Z}_\rho^2}$, (i.e., the map is b-normal; b-surjectivity is automatic here). This proves (a).

Part (b) also follows in part from a result in [KM15]. Indeed, the partial fiber diagonals are also binomial subvarieties of $\mathcal{Z}^{[3]}$, and near the corners formed by \mathcal{D} and \mathcal{X}_i are associated to further submonoids of (D.9) where $(n_i, m_i) = (n_j, m_j)$. Near corners formed by \mathcal{B} , they are associated to submonoids of $\sigma_{\mathcal{BB}} \cong \mathbb{N}^3 \ni (n_1, n_2, n_3)$ where $n_i = n_j$. The resolutions (D.10) and (D.11) are *compatible* with these submonoids, and by [KM15] Proposition 10.3 it follows that the diagonals lift to p-submanifolds of \mathcal{Z}_ρ^3 .

For (c), we consider the effect of the resolution $\mathcal{Z}_\rho^3 \longrightarrow \mathcal{Z}^{[3]}$ on the hypersurfaces of $\mathcal{Z}^{[3]}$. For example, $\mathcal{Z}^{[3]}$ has a boundary hypersurface given by the product $\mathcal{D} \times_{\mathcal{I}} \mathcal{D} \times_{\mathcal{I}} \mathcal{D}$. Considered as a binomial variety in its own right, this has smooth corners of codimension at most 3, and its monoidal complex is isomorphic to the quotient complex $\mathcal{P}_{\mathcal{Z}^{[3]}}/\sigma_{\mathcal{DD}}$. For instance, it is straightforward to verify that the quotient of (D.9) by $\sigma_{\mathcal{DD}} = \mathbb{N}\langle(1, 0, 1, 0, 1, 0)\rangle$ is a monoid freely generated by the images of \mathcal{AD}_i , \mathcal{DD}_j and \mathcal{DX}_k . The image of the resolution (D.10) under this quotient is likewise easily seen to coincide with the “total boundary blow-up” resolution (D.11) of \mathbb{N}^3 , which corresponds in turn to the sequence of blow-ups realizing the b-triple space $(\mathcal{D} \times_{\mathcal{I}} \mathcal{D} \times_{\mathcal{I}} \mathcal{D})_{\text{b}}$; thus $\mathcal{DD} \cong (\mathcal{D} \times_{\mathcal{I}} \mathcal{D} \times_{\mathcal{I}} \mathcal{D})_{\text{b}}$ in \mathcal{Z}_ρ^3 . A similar argument applies to \mathcal{AX}_i and to (c). \square

To define the triple space \mathcal{Z}_γ^3 , let Δ_{123} and Δ_{ij} , $1 \leq i < j \leq 3$ denote the maximal and partial diagonals, respectively, as submanifolds of \mathcal{Z}_ρ^3 , and for a boundary face F of \mathcal{Z}_ρ^3 write

$$\Delta_*^F = \Delta_* \cap F$$

for the intersection of the face with one of these diagonals. Then

$$\mathcal{Z}_\gamma^3 := [\mathcal{Z}_\rho^3, \Delta_{123}^{\mathcal{DD}}, \Delta_{123}^{\mathcal{BB}}, \Delta_{ij}^{\mathcal{DD}}, \Delta_{ij}^{\mathcal{BB}}, \Delta_{12}^{\mathcal{DX}}, \Delta_{13}^{\mathcal{DD}}, \Delta_{23}^{\mathcal{AD}}, \Delta_{12}^{\mathcal{BE}}, \Delta_{13}^{\mathcal{BB}}, \Delta_{23}^{\mathcal{AB}}]$$

In addition to the lifts of the boundary faces of \mathcal{Z}_ρ^3 , for which we use the same notation, \mathcal{Z}_γ^3 has as additional boundary faces the various front faces of the blow up, which we denote by

$$(\mathcal{DD}_{\text{sc}})_{123}, (\mathcal{BB}_{\text{sc}})_{123}, (\mathcal{DD}_{\text{sc}})_{ij}, (\mathcal{BB}_{\text{sc}})_{ij}, \\ \mathcal{DX}_{\text{sc}}^k, \mathcal{DD}_{\text{sc}}^j, \mathcal{AD}_{\text{sc}}^i, \mathcal{BE}_{\text{sc}}, \mathcal{BB}_{\text{sc}}, \mathcal{AB}_{\text{sc}}$$

where the superscripts indicate the corresponding face \mathcal{X}_i while the subscripts are used to indicate the diagonals.

Proposition D.6. (a) \mathcal{Z}_γ^3 admits three b-fibrations $\pi_{ij} : \mathcal{Z}_\gamma^3 \longrightarrow \mathcal{Z}_\gamma^2$ lifting the fiber projections $\pi_{ij} : \mathcal{Z}^{[3]} \longrightarrow \mathcal{Z}^{[2]}$.

- (b) *The partial diagonals meet all boundary hypersurfaces of \mathcal{Z}_γ^3 transversally.*
- (c) *The boundary face \mathcal{XX}_{iii} of \mathcal{Z}_γ^3 is isomorphic to the families scattering triple space $(\mathcal{X} \times_{\mathcal{I}} \mathcal{X} \times_{\mathcal{I}} \mathcal{X})_{\text{sc}}$ (with fiber X_{sc}^3), and the π_{ij} restrict over this face to the corresponding b-fibrations to $\mathcal{XX}_{ii} \cong (\mathcal{X} \times_U \mathcal{X})_{\text{sc}}$.*
- (d) *For $\varepsilon > 0$, the fiber $\mathcal{XX}_\varepsilon := \varrho^{-1}(\mathcal{I} \times \{\varepsilon\})$ of \mathcal{Z}_γ^3 is isomorphic to the scattering triple space $(\overline{\mathbb{R}}^3)_{\text{sc}}^3 \times \mathcal{I}$.*
- (e) *The face \mathcal{DD}_{sc} is diffeomorphic to a compactification of the fiber product ${}^\gamma\mathcal{TD} \times_{\mathcal{D}} {}^\gamma\mathcal{TD}$. The lifted projections π_{ij} map \mathcal{DD}_{sc} to $\mathcal{D}_{\text{sc}} \cong \overline{{}^\gamma\mathcal{TD}}$, and on the interior are identified with the linear maps $\pi_{12}(p, \xi, \xi') = (p, \xi)$, $\pi_{23}(p, \xi, \xi') = (p, \xi')$ and $\pi_{13}(p, \xi, \xi') = (p, \xi + \xi')$.*

The proof, which involves tedious local coordinate computations similar to those in the proof of Lemma D.3, is left as an exercise to the reader.

D.3. Pseudodifferential operators. Define ϱ -pseudodifferential operators by

$$\Psi_\varrho^{s, \mathcal{E}}(\mathcal{Z}) = \mathcal{A}^\mathcal{E} \mathcal{I}^s(\mathcal{Z}_\varrho^2; \Delta),$$

$$\mathcal{E} = (E_{\mathcal{D}}, E_{\mathcal{X}}, E_{\mathcal{B}}, E_{\mathcal{D}\mathcal{X}}, E_{\mathcal{X}\mathcal{D}}, E_{\mathcal{B}\mathcal{X}}, E_{\mathcal{B}\mathcal{D}}, \infty_{\mathcal{XX}_{ij}})$$

Here the index sets are indexed by the hypersurfaces of \mathcal{Z}_ϱ^2 ; $E_{\mathcal{X}\mathcal{X}}$ means a fixed index set for the union of hypersurfaces of the form \mathcal{XX}_{ii} , and likewise $E_{\mathcal{D}\mathcal{X}}$ means a fixed index sets for the union of hypersurfaces of the form \mathcal{DX}_i . We allow nontrivial asymptotics at all faces meeting Δ , as well as those spaces one step removed; however our kernels will be rapidly decreasing at the hypersurfaces two steps removed from Δ ; i.e., at \mathcal{XX}_{ij} for $i \neq j$. The extension to operators acting on sections of bundles $V_i \rightarrow \mathcal{Z}$, $i = 1, 2$ is achieved by considering coefficients in $\text{Hom}(\pi_L^* V_1, \pi_R^* V_2)$ on \mathcal{Z}_ϱ^2 .

By Lemma D.1.(b) and (d), The restriction of $Q \in \Psi_\varrho^{s, \mathcal{E}}(\mathcal{Z})$ to a boundary face $\mathcal{D} \cong (\mathcal{D} \times_{\mathcal{I}} \mathcal{D})_{\text{b}}$ or $\mathcal{XX}_{ii} \cong (\mathcal{X}_i \times_{\mathcal{I}} \mathcal{X}_i)_{\text{b}}$ (where restriction of a polyhomogeneous section to a boundary face is defined in general to be the restriction of the leading order term), or to a fiber $\mathcal{X}_\varepsilon^2 \cong (\mathcal{X}_\varepsilon \times_{\mathcal{I}} \mathcal{X}_\varepsilon)_{\text{b}}$ may be identified with the Schwartz kernel of a b-pseudodifferential operator on the associated space. Thus we define the *normal operators* of $Q \in \Psi_\varrho^{s, \mathcal{E}}(\mathcal{Z})$ by

$$N_{\mathcal{D}}(Q) := Q|_{\mathcal{D}} \in \Psi_{\text{b}}^{s, \mathcal{F}_{\mathcal{D}}}(\mathcal{D}/\mathcal{I}), \quad (\text{D.12a})$$

$$N_{\mathcal{X}_i}(Q) := Q|_{\mathcal{XX}_{ii}} \in \Psi_{\text{b}}^{s, \mathcal{F}_{\mathcal{X}_i}}(\mathcal{X}_i/\mathcal{I}), \quad (\text{D.12b})$$

$$N_{\mathcal{X}_\varepsilon}(Q) := Q|_{\mathcal{X}_\varepsilon^2} \in \Psi_{\text{b}}^{s, \mathcal{F}_{\mathcal{X}_\varepsilon}}(\mathcal{X}_\varepsilon/\mathcal{I}), \quad \varepsilon > 0 \quad (\text{D.12c})$$

where the index sets \mathcal{F}_* are determined from \mathcal{E} :

$$\begin{aligned}\mathcal{F}_{\mathcal{D}} &= (F_{\text{lf},\mathcal{X}}, F_{\text{rf},\mathcal{X}}, F_{\text{lf},\mathcal{B}}, F_{\text{rf},\mathcal{B}}, F_{\text{bf},\mathcal{X}}, F_{\text{bf},\mathcal{B}}) \\ &= (E_{\mathcal{X}\mathcal{D}}, E_{\mathcal{D}\mathcal{X}}, E_{\mathcal{B}\mathcal{Z}}, E_{\mathcal{Z}\mathcal{B}}, E_{\mathcal{X}\mathcal{X}}, E_{\mathcal{B}\mathcal{B}}) \\ \mathcal{F}_{\mathcal{X}_i} &= (F'_{\text{lf}}, F'_{\text{rf}}, F'_{\text{bf}}) = (E_{\mathcal{D}\mathcal{X}}, E_{\mathcal{X}\mathcal{D}}, E_{\mathcal{D}\mathcal{D}}) \\ \mathcal{F}_{\mathcal{X}_\varepsilon} &= (F''_{\text{lf}}, F''_{\text{rf}}, F''_{\text{bf}}) = (E_{\mathcal{Z}\mathcal{B}}, E_{\mathcal{B}\mathcal{Z}}, E_{\mathcal{B}\mathcal{B}})\end{aligned}$$

To define the action of $\Psi_\varrho^{s,\mathcal{E}}(\mathcal{Z})$ on $\dot{C}^\infty(\mathcal{Z})$ we need to be able to pushforward with respect to $\pi_L : \mathcal{Z}_\varrho^2 \rightarrow \mathcal{Z}$, which is only defined for fiber densities. Since it is most straightforward to prove mapping properties with respect to fiber b-densities, we define the action by

$$Qu := (\pi_L)_*(Q\pi_R^*u\pi_R^*\nu) \quad Q \in \Psi_\varrho^{s,\mathcal{E}}(\mathcal{Z}), \quad u \in \dot{C}^\infty(\mathcal{Z}),$$

where ν is the trivializing section of the μ density bundle on \mathcal{Z} obtained from the volume form of $g_\mu = (\rho_{\mathcal{D}}\rho_{\mathcal{B}})^2g$. In particular, $\pi_L^*\nu \otimes \pi_R^*\nu \otimes \pi_I^*(|\frac{d\varepsilon}{\varepsilon}|)$ is a trivializing fiber b-density on $\mathcal{Z}_\varrho^2 \rightarrow \mathcal{I}$.

Likewise, the composition of $A \in \Psi_\varrho^{s,\mathcal{E}}(\mathcal{Z})$ with $B \in \Psi_\varrho^{t,\mathcal{F}}(\mathcal{Z})$ is defined on the triple space by

$$A \circ B := (\pi_{13})_*(\pi_{12}^*A\pi_{23}^*B \otimes \pi_2^*\nu) \quad (\text{D.13})$$

where $\pi_{ij} : \mathcal{Z}_\varrho^3 \rightarrow \mathcal{Z}_\varrho^2$ denote the lifted projections to the double space and $\pi_i : \mathcal{Z}_\varrho^3 \rightarrow \mathcal{Z}$ denote the lifted projections to the single space \mathcal{Z} . The conditions under which this composition is well-defined are discussed in the Theorem below.

The properties of the ϱ -pseudodifferential operators that we shall need are summarized in Theorems D.7 and D.11

Theorem D.7.

- (a) Let $Q \in \Psi_\varrho^{s,\mathcal{E}}(\mathcal{Z})$. At the common boundary face $\mathcal{D} \cap \mathcal{X}_i$, the indicial operators of $N_{\mathcal{D}}(Q)$ and $N_{\mathcal{X}_i}(Q)$ are related by

$$I(N_{\mathcal{D}}(Q), \lambda) = I(N_{\mathcal{X}_i}(Q), -\lambda).$$

- (b) If $A \in \Psi_\varrho^{s,\mathcal{E}}(\mathcal{Z})$, $B \in \Psi_\varrho^{t,\mathcal{F}}(\mathcal{Z})$, and $E_{\mathcal{Z}\mathcal{B}} + F_{\mathcal{Z}\mathcal{B}} > 0$, then the composition $A \circ B \in \Psi_\varrho^{s+t,\mathcal{G}}(\mathcal{Z})$ is well-defined, with

$$\begin{aligned}G_{\mathcal{D}\mathcal{D}} &= (E_{\mathcal{D}\mathcal{D}} + F_{\mathcal{D}\mathcal{D}}) \overline{\cup} (E_{\mathcal{D}\mathcal{X}} + F_{\mathcal{D}\mathcal{X}}), & G_{\mathcal{X}\mathcal{D}} &= (E_{\mathcal{X}\mathcal{X}} + F_{\mathcal{X}\mathcal{X}}) \overline{\cup} (E_{\mathcal{X}\mathcal{D}} + F_{\mathcal{X}\mathcal{D}}), \\ G_{\mathcal{D}\mathcal{X}} &= (E_{\mathcal{D}\mathcal{X}} + F_{\mathcal{D}\mathcal{X}}) \overline{\cup} (E_{\mathcal{D}\mathcal{D}} + F_{\mathcal{D}\mathcal{D}}), & G_{\mathcal{X}\mathcal{X}} &= (E_{\mathcal{X}\mathcal{X}} + F_{\mathcal{X}\mathcal{X}}) \overline{\cup} (E_{\mathcal{X}\mathcal{D}} + F_{\mathcal{X}\mathcal{D}}), \\ G_{\mathcal{B}\mathcal{Z}} &= (E_{\mathcal{B}\mathcal{B}} + F_{\mathcal{B}\mathcal{B}}) \overline{\cup} E_{\mathcal{B}\mathcal{Z}}, & G_{\mathcal{Z}\mathcal{B}} &= F_{\mathcal{Z}\mathcal{B}} \overline{\cup} (E_{\mathcal{Z}\mathcal{B}} + F_{\mathcal{Z}\mathcal{B}}), \\ G_{\mathcal{B}\mathcal{B}} &= (E_{\mathcal{B}\mathcal{B}} + F_{\mathcal{B}\mathcal{B}}) \overline{\cup} (E_{\mathcal{B}\mathcal{Z}} + F_{\mathcal{B}\mathcal{Z}}).\end{aligned}$$

We always have

$$N_{\mathcal{X}_\varepsilon}(A \circ B) = N_{\mathcal{X}_\varepsilon}(A) \circ N_{\mathcal{X}_\varepsilon}(B) \in \Psi_{\text{b}}^{s+t,*}(\mathcal{X}_\varepsilon/\mathcal{I}),$$

and, provided that $E_{\mathcal{D}\mathcal{X}} + F_{\mathcal{X}\mathcal{D}} > E_{\mathcal{D}\mathcal{D}} + F_{\mathcal{D}\mathcal{D}} \geq 0$ and $E_{\mathcal{X}\mathcal{D}} + F_{\mathcal{D}\mathcal{X}} > E_{\mathcal{X}\mathcal{X}} + F_{\mathcal{X}\mathcal{X}} \geq 0$, respectively, then

$$\begin{aligned} N_{\mathcal{D}}(A \circ B) &= N_{\mathcal{D}}(A) \circ N_{\mathcal{D}}(B) \in \Psi_{\mathbf{b}}^{s+t,*}(\mathcal{D}/\mathcal{I}), \\ N_{\mathcal{X}_i}(A \circ B) &= N_{\mathcal{X}_i}(A) \circ N_{\mathcal{X}_i}(B) \in \Psi_{\mathbf{b}}^{s+t,*}(\mathcal{X}_i/\mathcal{I}), \end{aligned} \quad (\text{D.14})$$

(c) *There is an injective homomorphism of graded algebras*

$$\text{Diff}_{\rho}^*(\mathcal{Z}) \hookrightarrow \Psi_{\rho}^*(\mathcal{Z}),$$

with respect to which $\text{Diff}_{\rho}^k(\mathcal{Z}) \subset \Psi_{\rho}^{k,(0\mathcal{D},0\mathcal{X},0\mathcal{B},\infty^*)}(\mathcal{Z})$, and the normal operators of $P \in \text{Diff}_{\rho}^*(\mathcal{Z})$ are identified with restriction of P to the corresponding boundary face of \mathcal{Z} :

$$N_{\mathcal{D}}(P) \cong P|_{\mathcal{D}}, \quad N_{\mathcal{X}_i}(P) \cong P|_{\mathcal{X}_i}, \quad N_{\mathcal{X}_{\varepsilon}}(P) \cong P|_{\mathcal{X}_{\varepsilon}}.$$

Proof. To prove (a) we employ a local coordinate description. Near the interior of $\mathcal{D}\mathcal{D} \cap \mathcal{X}\mathcal{X}_{ii}$ in \mathcal{Z}_{ρ}^2 , the coordinates

$$(x, r', s, y, y', q), \quad s = \frac{x'}{x} = \frac{r}{r'}$$

are valid, with x boundary defining for $\mathcal{D}\mathcal{D}$ and r' boundary defining for $\mathcal{X}\mathcal{X}_{ii}$. (Alternatively, we may use $(x', r, s^{-1}, y, y', q)$). Here (x, r, y, q) denote local coordinates on \mathcal{Z} pulled back from the right and (x', r', y', q) denote coordinates pulled back from the left. In any case, $I(N_{\mathcal{D}}(Q), \lambda)$ may be expressed as the Mellin transform of the restriction of Q to $\{x = r' = 0\}$ with respect to s , while $I(N_{\mathcal{D}}(Q), \lambda)$ is the Mellin transform of the same with respect to s^{-1} , from which (a) follows.

The composition of pseudodifferential operators in (b) is defined by (D.13). By wavefront considerations, it is clear that the only interior conormal singularities of $A \circ B \in \mathcal{Z}_{\rho}^2$ can occur along the fiber diagonal, and the fact that $A \circ B$ has interior conormal order $s + t$ (with the principal symbolic composition formula $\sigma(A \circ B) = \sigma(A)\sigma(B)$) follows from the usual local considerations. To verify the index set formulae, we may assume $s = t = -\infty$. The result is then a consequence of the pullback and pushforward theorems for polyhomogeneous functions with respect to b-fibrations [Mel92]. To invoke these theorems, it is only necessary to determine the mapping properties of boundary hypersurfaces with respect to the lifted projections π_{ij} (since the boundary exponents occurring in the b-fibrations $\pi_{ij} : \mathcal{Z}_{\rho}^3 \rightarrow \mathcal{Z}_{\rho}^2$ are either 0 or 1), but these have been made obvious from the notation. For instance, the formula for $G_{\mathcal{D}\mathcal{X}}$ is a consequence of the following:

$$\begin{aligned} \pi_{13}^{-1}(\mathcal{D}\mathcal{X}) &= \mathcal{D}\mathcal{D}\mathcal{X} \cup \mathcal{D}\mathcal{X}\mathcal{X}, \\ (\pi_{12})_{\#}(\mathcal{D}\mathcal{D}\mathcal{X}) &= \mathcal{D}\mathcal{D}, \quad (\pi_{23})_{\#}(\mathcal{D}\mathcal{D}\mathcal{X}) = \mathcal{D}\mathcal{X}, \\ (\pi_{12})_{\#}(\mathcal{D}\mathcal{X}\mathcal{X}) &= \mathcal{D}\mathcal{X}, \quad (\pi_{23})_{\#}(\mathcal{D}\mathcal{X}\mathcal{X}) = \mathcal{X}\mathcal{X}. \end{aligned}$$

The others are similar.

To see (D.14), observe that if $E_{\mathcal{D}}, F_{\mathcal{D}} \geq 0$ and $E_{\mathcal{D}\mathcal{X}} + F_{\mathcal{D}\mathcal{X}} > 0$, then the leading order contribution of $A \circ B$ at \mathcal{D} comes from $\mathcal{D}\mathcal{D}$. In other words, this leading order term is given by

$$(\pi_{13})_*(\pi_{12}^* A \pi_{23}^* B \pi_2^* \nu_{\mathcal{D}}),$$

$$\pi_{ij} : \mathcal{D}\mathcal{D} \longrightarrow \mathcal{D} \subset \mathcal{Z}_{\varrho}^2, \quad \pi_2 : \mathcal{D}\mathcal{D} \longrightarrow \mathcal{D} \subset \mathcal{Z},$$

(Note that the fiberwise b-density $\pi_2^* \nu$ on \mathcal{Z}_{ϱ}^3 restricts canonically to a fiberwise b-density on the hypersurface $\mathcal{D}\mathcal{D}$, which is intertwined with the pullback of the restriction of ν to \mathcal{D} .) That this may be identified with the usual composition of b-pseudodifferential operators then follows from Lemma D.5. A similar argument applies to the composition of normal operators with respect to \mathcal{X}_i and $\mathcal{X}_{\varepsilon}$.

For part (c), let $P \in \text{Diff}_{\varrho}^k(\mathcal{Z})$. By Lemma D.2, the lifts $\pi_L^*(P)$ and $\pi_R^*(P)$ differentiate transversally to Δ and tangentially to all boundary faces of \mathcal{Z}_{ϱ}^2 , and the image of P in $\Psi_{\varrho}^{k,*}(\mathcal{Z})$ is given by applying $\pi_L^*(P)$ to the kernel of $\text{Id} \in \Psi_{\varrho}^{0,(0,\mathcal{D},0\mathcal{X},0\mathcal{B})}(\mathcal{Z})$. Composing two such images of $P_1, P_2 \in \text{Diff}_{\varrho}^*(\mathcal{Z})$ in Ψ_{ϱ}^* is readily seen to be equivalent to the image of $P_1 \circ P_2$. The identification of the normal operators follows from Lemma D.2. \square

Define γ -pseudodifferential operators by

$$\Psi_{\gamma}^{s,\mathcal{E}}(\mathcal{Z}) = \mathcal{A}^{\mathcal{E}'} \mathcal{I}^s(\mathcal{Z}_{\gamma}^2; \Delta),$$

$$\mathcal{E} = (E_{\mathcal{D}_{\text{sc}}}, E_{\mathcal{X}}, E_{\mathcal{B}_{\text{sc}}}, \infty_*).$$

$$\mathcal{E}' = (E_{\mathcal{D}_{\text{sc}}} - 3, E_{\mathcal{X}}, E_{\mathcal{B}_{\text{sc}}} - 3, \infty_*).$$

These kernels are required to have rapid decay at all boundary hypersurfaces besides those which meet the lift of Δ . The shift in index by -3 at \mathcal{D}_{sc} and \mathcal{B}_{sc} may be regarded as a normalization convention; in particular $\text{Id} \in \Psi_{\gamma}^{0,(0,0,0)}(\mathcal{Z})$ with this convention. (The lift of the delta function—which is homogeneous of degree -1 as a distribution—of the fiber diagonal from \mathcal{Z}_{ϱ}^2 to \mathcal{Z}_{γ}^2 has polyhomogeneous order -3 at \mathcal{D}_{sc} and \mathcal{B}_{sc} .) Likewise, the pullback of the fiber b-density bundle $\Omega_{\text{b}}(\mathcal{Z}_{\varrho}^2)$ to \mathcal{Z}_{γ}^2 is isomorphic to $\rho_{\mathcal{D}_{\text{sc}}}^3 \rho_{\mathcal{B}_{\text{sc}}}^3 \Omega_{\text{b}}(\mathcal{Z}_{\gamma}^2)$, so the shift is nullified when these Schwartz kernels are multiplied by the volume form ν from \mathcal{Z} prior to pushing forward.

By Lemma D.3, the restriction of $Q \in \Psi_{\gamma}^{s,\mathcal{E}}(\mathcal{Z})$ to a boundary face \mathcal{X}_{ii} may be identified with the kernel of a family of scattering pseudodifferential operators on \mathcal{X}_i , and likewise the restriction to the fiber $\mathcal{X}_{\varepsilon}^2 = \varrho^{-1}(\mathcal{I} \times \{\varepsilon\})$, $\varepsilon > 0$ may be identified with a family of scattering pseudodifferential operators on $\mathcal{X}_{\varepsilon}$. The restriction of Q to \mathcal{D}_{sc} defines a conormal distribution on $\overline{\gamma T\mathcal{D}}$ conormal to the 0-section, whose fiberwise Fourier transform may

be identified with a symbol. Thus we define

$$\sigma_{\mathcal{D}}(Q) = \mathcal{F}_{\text{fib}}(Q|_{\mathcal{D}_{\text{sc}}}) \in \mathcal{A}^{\mathcal{F}_{\mathcal{D}}} S_{1,0}^s(\gamma T^* \mathcal{D}), \quad (\text{D.15a})$$

$$N_{\mathcal{X}_i}(Q) = Q|_{\mathcal{X}_{ii}} \in \Psi_{\text{sc}}^{s,f}(\mathcal{X}_i/\mathcal{I}) \quad (\text{D.15b})$$

$$N_{\mathcal{X}_\varepsilon}(Q) = Q|_{\mathcal{X}_\varepsilon^2} \in \Psi_{\text{sc}}^{s,f'}(\mathcal{X}_\varepsilon/\mathcal{I}), \quad \varepsilon > 0 \quad (\text{D.15c})$$

where

$$\mathcal{F}_{\mathcal{D}} = (E_{\mathcal{B}_{\text{sc}}}, E_{\mathcal{X}}), \quad f = E_{\mathcal{D}_{\text{sc}}}, \quad f' = E_{\mathcal{B}_{\text{sc}}}$$

Theorem D.8.

- (a) Let $Q \in \Psi_{\gamma}^{s,\mathcal{E}}(\mathcal{Z})$. At the common boundary face $\mathcal{D} \cap \mathcal{X}_i$, the symbol $\sigma_{\mathcal{D}}(Q)$ and the scattering symbol $\sigma_{\text{sc}}(N_{\mathcal{X}_i}(Q))$ agree.
- (b) If $A \in \Psi_{\gamma}^{s,\mathcal{E}}(\mathcal{Z})$, $B \in \Psi_{\gamma}^{t,\mathcal{F}}(\mathcal{Z})$ then the composition $A \circ B \in \Psi_{\rho}^{s+t,\mathcal{G}}(\mathcal{Z})$ is well-defined, with

$$G_{\mathcal{D}_{\text{sc}}} = E_{\mathcal{D}_{\text{sc}}} + F_{\mathcal{D}_{\text{sc}}} \quad G_{\mathcal{B}_{\text{sc}}} = E_{\mathcal{B}_{\text{sc}}} + F_{\mathcal{B}_{\text{sc}}} \quad G_{\mathcal{X}} = E_{\mathcal{X}} + F_{\mathcal{X}}.$$

and the normal operator maps are homomorphisms:

$$\begin{aligned} N_{\mathcal{X}_\varepsilon}(A \circ B) &= N_{\mathcal{X}_\varepsilon}(A) \circ N_{\mathcal{X}_\varepsilon}(B) \in \Psi_{\text{sc}}^{s+t,*}(\mathcal{X}_\varepsilon/\mathcal{I}), \\ N_{\mathcal{X}_i}(A \circ B) &= N_{\mathcal{X}_i}(A) \circ N_{\mathcal{X}_i}(B) \in \Psi_{\text{sc}}^{s+t,*}(\mathcal{X}_i/\mathcal{I}), \\ \sigma_{\mathcal{D}}(A \circ B) &= \sigma_{\mathcal{D}}(A) \sigma_{\mathcal{D}}(B) \in \mathcal{A}^* S_{1,0}^{s+t}(\gamma T^* \mathcal{D}). \end{aligned} \quad (\text{D.16})$$

- (c) There is an injective homomorphism of graded algebras

$$\text{Diff}_{\gamma}^*(\mathcal{Z}) \hookrightarrow \Psi_{\gamma}^*(\mathcal{Z}),$$

with respect to which $\text{Diff}_{\gamma}^k(\mathcal{Z}) \subset \Psi_{\gamma}^{k,(0_{\mathcal{D}_{\text{sc}}}, 0_{\mathcal{X}}, 0_{\mathcal{B}_{\text{sc}}})}(\mathcal{Z})$, and the normal operators of $P \in \text{Diff}_{\gamma}^*(\mathcal{Z})$ are identified with restriction of P to the corresponding boundary face of \mathcal{Z} :

$$N_{\mathcal{X}_i}(P) \cong P|_{\mathcal{X}_i}, \quad N_{\mathcal{X}_\varepsilon}(P) \cong P|_{\mathcal{X}_\varepsilon}.$$

Likewise, the semiclassical symbol of P is the same, considered as a differential or pseudodifferential operator.

The proof is similar to the proof of Theorem D.7, and is left to the reader. It is worth remarking that, although the triple space \mathcal{Z}_{γ}^3 is more complex, the composition result here is much simpler owing to the rapid decay of elements of $\Psi_{\gamma}^*(\mathcal{Z})$ away from all boundary faces except \mathcal{D}_{sc} , \mathcal{B}_{sc} and \mathcal{X} .

D.4. Sobolev spaces and mapping properties. Consider the set of *small* ρ -pseudodifferential operators:

$$\Psi_{\rho,\text{sm}}^s(\mathcal{Z}) := \Psi_{\rho}^{s,(\geq 0_{\mathcal{D}}, \geq 0_{\mathcal{X}}, \geq 0_{\mathcal{B}}, \infty_*)}(\mathcal{Z}),$$

where the notation indicates that the index sets of the operators have leading order $(0, 0)$ at \mathcal{D} , \mathcal{X} and \mathcal{B} and are empty everywhere else. By Theorem D.7 this set is closed with respect to composition. Furthermore, if $A \in \Psi_{\rho,\text{sm}}^s(\mathcal{Z})$ is elliptic (meaning its principal symbol as a conormal distribution

is uniformly invertible off the diagonal), then there exists a *small parametrix*, meaning $B \in \Psi_{\varrho, \text{sm}}^{-s}(\mathcal{Z})$ such that $I - AB, I - BA \in \Psi_{\varrho, \text{sm}}^{-\infty}(\mathcal{Z})$.

The (fiberwise) ϱ Sobolev spaces are most efficiently defined as follows. Fix a fiber Z of $\mu : \mathcal{Z} \rightarrow \mathcal{I}$ and set

$$H_{\varrho}^s(Z) = \{u \in C^{-\infty}(Z) : Au \in L^2(Z; \bar{g}), \forall A \in \Psi_{\varrho, \text{sm}}^s(Z)\}$$

where $\Psi_{\varrho, \text{sm}}^s(Z)$ denotes the restriction of $\Psi_{\varrho, \text{sm}}^s(\mathcal{Z})$ to the corresponding fiber of $\mu : \mathcal{Z}^2 \rightarrow \mathcal{I}$. It will follow from the results below that if $s \geq 0$, this can be taken to be a domain in $L^2(Z)$. Moreover, if $s \in \mathbb{N}$ then the A can be taken in $\text{Diff}_{\varrho}^s(Z)$.

Lemma D.9.

- (a) Every $A \in \Psi_{\varrho, \text{sm}}^0(Z)$ extends to a bounded operator $A : L^2(Z; \bar{g}) \rightarrow L^2(Z; \bar{g})$. In particular, $H_{\varrho}^0(Z; \bar{g}) \equiv L^2(Z; \bar{g})$.
- (b) If $A \in \Psi_{\varrho, \text{sm}}^{-\infty}(Z)$, then

$$A : L^2(Z) \rightarrow H_{\varrho}^s(Z), \quad \forall s \in \mathbb{R}.$$

- (c) Fix any elliptic element $P \in \Psi_{\varrho, \text{sm}}^s(Z)$. Then

$$H_{\varrho}^s(Z) = \{u \in C^{-\infty}(Z) : Pu \in L^2(Z)\}.$$

Proof. The first result follows by a standard trick due to Hörmander. Namely, let $c > 0$ such that $c^2 \geq \sup |\sigma(A)^* \sigma(A)|$. Then by an iterative symbolic procedure there exists a formally self-adjoint $B \in \Psi_{\varrho}^s(Z)$ and $R \in \Psi_{\varrho, \text{sm}}^{-\infty}(Z)$ such that

$$B^2 = c^2 \text{Id} - A^* A + R$$

with composition here defined as operators on distributions. Then for $u \in C_c^{\infty}(Z)$,

$$\|Au\|_{L^2}^2 = \langle c^2 u, u \rangle + \langle Ru, u \rangle - \|Bu\|^2 \leq c \|u\|^2 + \langle Ru, u \rangle.$$

Boundedness then follows from part (b) with $s = 0$, which follows in turn from Schur's Lemma.

Indeed, any $R \in \Psi_{\varrho, \text{sm}}^{-\infty}(Z)$ is represented by a kernel on Z_{ϱ}^2 which is uniformly bounded (by the hypothesis that its index sets are ≥ 0) and smooth on the interior. Schur's Lemma states that this extends to a bounded operator from $L^2(Z; \bar{g})$ to $L^2(Z; \bar{g})$ provided its left and right projections $(\pi_L)_*(|R|), (\pi_R)_*(|R|) \in \mathcal{A}^*(Z)$ are uniformly bounded. Here the pushforward is with respect to the volume form associated to \bar{g} ; to convert to the natural b-volume form on Z we consider instead the conjugated operator $\tilde{R} = \rho^{-3/2} R \rho^{3/2}$, which has the same index sets as R , namely $\tilde{R} \in \mathcal{A}^{(0\mathcal{D}, 0\mathcal{X}, 0\mathcal{B}, \infty^*)}(Z_{\varrho}^2)$. By the pushforward and pullback theorems in [Mel92], this has left and right projections in $\mathcal{A}^0(Z)$, which are indeed uniformly bounded. This proves part (b) in the case $s = 0$ (and hence part (a)). The general case follows by composing with any $A \in \Psi_{\varrho, \text{sm}}^s(Z)$ and using the fact that $\Psi_{\varrho, \text{sm}}^{-\infty}$ is an ideal.

From part (a) and composition, it follows that $\Psi_{\varrho, \text{sm}}^{-s} \ni Q : L^2 \longrightarrow H_{\varrho}^s$, and then part (c) follows from the existence of a parametrix and the identity $QPu = u - Ru$, $R \in \Psi_{\varrho, \text{sm}}^{-\infty}$. \square

For any choice of elliptic operator $P_s \in \Psi_{\varrho, \text{sm}}^{s/2}(Z)$,

$$P_s^* P_s + 1 : H_{\varrho}^s(Z) \longrightarrow L^2(Z)$$

is bounded, self-adjoint, and easily seen to have no nullspace. It is therefore an isomorphism, in terms of which $H_{\varrho}^s(Z)$ may be given the structure of a complete Hilbert space, with topology independent of the choice of P_s . The following result follows from this observation and composition.

Corollary D.10. *Every $A \in \Psi_{\varrho, \text{sm}}^s(Z)$ extends to a bounded operator*

$$A : H_{\varrho}^m(Z) \longrightarrow H_{\varrho}^{m-s}(Z), \quad \forall m \in \mathbb{R}.$$

By definition, an operator $A \in \Psi_{\varrho}^*(\mathcal{Z})$ is smoothly parameterized by \mathcal{I} , i.e., A can be viewed as a family of pseudodifferential operators with respect to the fibration $\mu : \mathcal{Z} \longrightarrow \mathcal{I}$. It follows that the space of smooth sections of the Hilbert bundle over \mathcal{I} , with fibers $H_{\varrho}^s(Z)$, is characterized as follows:

$$C^\infty(\mathcal{I}; H_{\varrho}^s(Z)) = \{u : Au \in C^\infty(\mathcal{I}; L^2(Z)), \text{ for all } A \in \Psi_{\varrho, \text{sm}}^s(Z)\}$$

where “for all” may be replaced by “for some elliptic”.

Finally, we consider the mapping properties of the full calculus $\Psi_{\varrho}^*(\mathcal{Z})$ with respect to the spaces $C^\infty(\mathcal{I}; H_{\mu, \varrho}^{m, s}(Z))$, where we restrict to $m \in \mathbb{N}$ for convenience (to avoid discussion of the associated pseudodifferential operators).

Theorem D.11. *If $Q \in \Psi_{\varrho}^{s, \mathcal{E}}(\mathcal{Z})$ and \mathcal{E} satisfies*

$$\begin{aligned} E_{\mathcal{X}\mathcal{X}} &\geq 0, & E_{\mathcal{T}\mathcal{D}}, E_{\mathcal{B}\mathcal{B}} &\geq \alpha' - \alpha, \\ E_{\mathcal{T}\mathcal{X}}, E_{\mathcal{B}\mathcal{Z}} &> \alpha' + \frac{3}{2}, & E_{\mathcal{X}\mathcal{D}}, E_{\mathcal{Z}\mathcal{B}} &> -\alpha - \frac{3}{2}, \end{aligned}$$

then Q extends to a bounded operator

$$Q : C^\infty(\mathcal{I}; \rho^\alpha H_{\mu, \varrho}^{k, l}(Z)) \longrightarrow C^\infty(\mathcal{I}; \rho^{\alpha'} H_{\mu, \varrho}^{k, l-s}(Z)). \quad (\text{D.17})$$

for any $k \in \mathbb{N}$ and $l \in \mathbb{R}$, where $\rho = \rho_{\mathcal{D}} \rho_{\mathcal{B}}$.

Proof. We first restate the result in terms of b-metrics; since $L^2(Z; \bar{g}) = \rho^{3/2} L^2(Z; g_b)$, the result to prove is equivalent to the boundedness of

$$Q : C^\infty(\mathcal{I}; \rho^\beta H_{\mu, \varrho}^{k, l}(Z; g_b)) \longrightarrow C^\infty(\mathcal{I}; \rho^{\beta'} H_{\mu, \varrho}^{k, l-s}(Z; g_b)),$$

under the assumptions that

$$\begin{aligned} E_{\mathcal{X}\mathcal{X}} &\geq 0, & E_{\mathcal{T}\mathcal{D}}, E_{\mathcal{B}\mathcal{B}} &\geq \beta' - \beta, \\ E_{\mathcal{T}\mathcal{X}}, E_{\mathcal{B}\mathcal{Z}} &> \beta', & E_{\mathcal{X}\mathcal{D}}, E_{\mathcal{Z}\mathcal{B}} &> -\beta, \end{aligned}$$

which is in turn equivalent to boundedness of

$$\begin{aligned} \rho^{-\beta'} Q \rho^\beta : C^\infty(\mathcal{I}; H_{\mu, \varrho}^{k, l}(\mathcal{Z}; g_b)) &\longrightarrow C^\infty(\mathcal{I}; H_{\mu, \varrho}^{k, l-s}(\mathcal{Z}; g_b)), \\ \rho^{-\beta'} Q \rho^\beta &\in \Psi_{\varrho}^{s, \mathcal{E}'}(\mathcal{Z}), \\ E'_{\mathcal{D}} &= E_{\mathcal{D}} + (\beta - \beta'), \quad E'_{\mathcal{B}} = E_{\mathcal{B}} + (\beta - \beta'), \\ E'_{\mathcal{D}'} &= E_{\mathcal{D}'} - \beta', \quad E'_{\mathcal{X}} = E_{\mathcal{X}} + \beta, \\ E'_{\mathcal{B}'} &= E_{\mathcal{B}'} - \beta', \quad E'_{\mathcal{Z}} = E_{\mathcal{Z}} + \beta, \\ E'_{\mathcal{X}'} &= E_{\mathcal{X}'}, \end{aligned}$$

so it suffices to consider the case where $\beta = \beta' = 0$.

Likewise, since everything is smoothly parameterized by \mathcal{I} , it suffices to restrict Q to a fixed fiber $Z_\varrho^2 = \mu^{-1}(q)$.

We first consider the case $k = 0$. Q may be decomposed as

$$Q = Q_{\text{sm}} + Q_\infty, \quad Q_{\text{sm}} \in \Psi_{\varrho, \text{sm}}^s(Z) \quad Q_\infty \in \Psi_\varrho^{-\infty, \mathcal{E}}(Z),$$

into an element of the small calculus and a smoothing element, which may be considered separately. Then Q_{sm} was shown above to be bounded, and Q_∞ is seen to be bounded by an application of Schur's Lemma. Indeed,

$$(\pi_R)_*(|Q| \pi_L^* \nu) \in \mathcal{A}^{\mathcal{F}_R}(Z), \quad (\pi_L)_*(|Q| \pi_R^* \nu) \in \mathcal{A}^{\mathcal{F}_L}(Z)$$

are well-defined provided $E_{\mathcal{B}'} > 0$ (respectively $E_{\mathcal{Z}} > 0$), and

$$\begin{aligned} \mathcal{F}_R &= (F_{\mathcal{D}}, F_{\mathcal{X}}, F_{\mathcal{B}}) = (E_{\mathcal{D}} \bar{\cup} E_{\mathcal{X}}, E_{\mathcal{X}} \bar{\cup} E_{\mathcal{D}'}, E_{\mathcal{B}} \bar{\cup} E_{\mathcal{Z}}), \\ \mathcal{F}_L &= (F'_{\mathcal{D}}, F'_{\mathcal{X}}, F'_{\mathcal{B}}) = (E_{\mathcal{D}} \bar{\cup} E_{\mathcal{D}'}, E_{\mathcal{X}} \bar{\cup} E_{\mathcal{X}'}, E_{\mathcal{B}} \bar{\cup} E_{\mathcal{B}'}). \end{aligned}$$

The hypotheses on \mathcal{E} guarantee that these index sets are ≥ 0 , hence the pushforwards are uniformly bounded.

In the case $k > 0$, observe that a general vector field in $\mathcal{V}_\mu(Z) \equiv \mathcal{V}_b(Z)$ may be decomposed into an element of $\mathcal{V}_\varrho(Z)$ and a multiple of $\varepsilon \partial_\varepsilon$, where the latter denotes (by abuse of notation) a choice of lift of the canonical b-vector field $\varepsilon \partial_\varepsilon$ on $[0, 1)$. Since $\varepsilon \partial_\varepsilon$ differentiates in the fiber direction, $\varepsilon \partial_\varepsilon(Qu) = (\varepsilon \partial_\varepsilon Q)u + Q(\varepsilon \partial_\varepsilon u)$, but $\varepsilon \partial_\varepsilon Q \in \Psi_\varrho^{s, \mathcal{E}}(Z)$ again since $\varepsilon \partial_\varepsilon$ is tangent to all boundary faces of Z_ϱ^2 . The general result then follows by commutation and induction. \square

Proceeding in a similar manner, we may characterize the fiberwise γ Sobolev spaces by

$$H_\gamma^s(Z) = \{u \in C^{-\infty}(Z) : Au \in L^2(Z; \bar{g}), \forall / (\exists \text{ elliptic}) A \in \Psi_{\gamma, \text{sm}}^s(Z)\},$$

where $\Psi_{\gamma, \text{sm}}^s(Z)$ denotes the restriction to a fiber Z_γ^2 of $\mu : \mathcal{Z}_\gamma^2 \longrightarrow \mathcal{I}$ of the set $\Psi_{\gamma, \text{sm}}^s = \Psi_\gamma^{s, \mathcal{E}}(\mathcal{Z})$ where $E_{\mathcal{D}_{\text{sc}}}, E_{\mathcal{B}_{\text{sc}}}, E_{\mathcal{X}} \geq 0$. A nearly identical proof leads to the obvious analogue of Corollary D.10, and we have

Theorem D.12. *If $Q \in \Psi_\gamma^{s,\mathcal{E}}(\mathcal{Z})$ and \mathcal{E} satisfies*

$$E_{\mathcal{X}} \geq 0, \quad E_{\mathcal{D}_{\text{sc}}}, E_{\mathcal{B}_{\text{sc}}} \geq \alpha' - \alpha,$$

then Q extends to a bounded operator

$$Q : C^\infty(\mathcal{I}; \rho^\alpha H_{\mu,\varrho,\gamma}^{k,l,m}(Z)) \longrightarrow C^\infty(\mathcal{I}; \rho^{\alpha'} H_{\mu,\varrho,\gamma}^{k,l,m-s}(Z)). \quad (\text{D.18})$$

for any $k, l \in \mathbb{N}$ and $m \in \mathbb{R}$.

Proof. The proof is almost entirely similar to the proof of Theorem D.11 above. The only additional wrinkle is that, for nonzero k, l , it is necessary to consider the lift to \mathcal{Z}_γ^2 of vector fields in $\mathcal{V}_\varrho(\mathcal{Z})$. As shown above, these are tangent to the boundary faces of \mathcal{Z}_ϱ^2 , so it follows that they lift to be singular at \mathcal{D}_{sc} and \mathcal{B}_{sc} in \mathcal{Z}_γ^2 . More precisely, the lifts of $V \in \mathcal{V}_\varrho(\mathcal{Z})$ have the form

$$\pi_R^*(V) = (\rho_{\mathcal{D}_{\text{sc}}})^{-1}(\rho_{\mathcal{B}_{\text{sc}}})^{-1}\tilde{V}, \quad \pi_L^*(V) = (\rho_{\mathcal{D}_{\text{sc}}})^{-1}(\rho_{\mathcal{B}_{\text{sc}}})^{-1}\tilde{V}',$$

where \tilde{V} and \tilde{V}' restrict to the *opposite* fiberwise constant vector field on \mathcal{D}_{sc} and \mathcal{B}_{sc} , as follows from the computations in the proof of Lemma D.3. In particular it follows that the order $(\rho_{\mathcal{D}_{\text{sc}}}\rho_{\mathcal{B}_{\text{sc}}})^{-1}$ term of the commutator $[Q, V] = (\pi_L^*(V) + \pi_R^*(V))Q$ (the sign change on $\pi_R^*(V)$ is due to integration by parts) vanishes, so that

$$[Q, V] \in \Psi_\gamma^{s,\mathcal{E}}(\mathcal{Z}), \quad V \in \mathcal{V}_\varrho(\mathcal{Z}).$$

This may be iterated to show that, for $P \in \text{Diff}_\varrho^l(\mathcal{Z})$, $PQ - QP' \in \Psi_\gamma^{s,\mathcal{E}}$ for some $P' \in \text{Diff}_\varrho^l(\mathcal{Z})$, from which the result follows.

Finally, for $k \neq 0$, it suffices to note that $\varepsilon\partial_\varepsilon Q \in \Psi_\gamma^{s,\mathcal{E}}(\mathcal{Z})$. Then the lift any $V \in \mathcal{V}_\varrho(\mathcal{Z})$ can be locally decomposed into $\varepsilon\partial_\varepsilon$ and the lift of an element in $\mathcal{V}_\varrho(\mathcal{Z})$. Since $\varepsilon\partial_\varepsilon(Qu) = (\varepsilon\partial_\varepsilon Q)u + Q(\varepsilon\partial_\varepsilon u)$,

$$[Q, P] \in \Psi_\gamma^{-\infty,\mathcal{E}}(\mathcal{Z}), \quad P \in \text{Diff}_\varrho^k(\mathcal{Z}),$$

and the result for general k and l follows. \square

D.5. Residual ideals. Consider the subset $\Psi_\varrho^{s,(\infty*)}(\mathcal{Z}) \subset \Psi_\varrho^{s,*}(\mathcal{Z})$. These *residual operators* have kernels which are conormal of order s at the diagonal and vanish rapidly at all boundary faces of \mathcal{Z}_ϱ^2 . They lift to similarly conormal kernels with rapid vanishing on the space \mathcal{Z}_γ^2 , which is to say the subset $\Psi_\gamma^{s,(\infty*)}(\mathcal{Z}) \subset \Psi_\gamma^{s,*}(\mathcal{Z})$, and conversely $\Psi_\gamma^{s,(\infty*)}(\mathcal{Z})$ pushes forward under the blow-down $\mathcal{Z}_\gamma^2 \longrightarrow \mathcal{Z}_\varrho^2$ to $\Psi_\varrho^{s,(\infty*)}(\mathcal{Z})$.

We identify these two subspaces and denote them simply by

$$\rho^\infty \Psi^s(\mathcal{Z}) := \Psi_\varrho^{s,(\infty*)}(\mathcal{Z}) \equiv \Psi_\gamma^{s,(\infty*)}(\mathcal{Z}), \quad s \in \mathbb{R}.$$

It follows from Theorems D.7 and D.8 that these subsets form graded ideals with respect to composition, i.e.,

$$\begin{aligned} \rho^\infty \Psi^s(\mathcal{Z}) \circ \Psi_\varrho^{t,\mathcal{E}}(\mathcal{Z}), \quad \Psi_\varrho^{t,\mathcal{E}}(\mathcal{Z}) \circ \rho^\infty \Psi^s(\mathcal{Z}) &\subset \rho^\infty \Psi^{s+t}(\mathcal{Z}), \\ \rho^\infty \Psi^s(\mathcal{Z}) \circ \Psi_\gamma^{t,\mathcal{E}}(\mathcal{Z}), \quad \Psi_\gamma^{t,\mathcal{E}}(\mathcal{Z}) \circ \rho^\infty \Psi^s(\mathcal{Z}) &\subset \rho^\infty \Psi^{s+t}(\mathcal{Z}). \end{aligned}$$

(c.f. (C.20)). We refer to $\bigcup_s \rho^\infty \Psi^s(\mathcal{Z})$ as the *residual ideal*.

D.6. Parametrices. Finally, we employ the pseudodifferential operator calculus developed above to construct parametrices for the linearized Coulomb gauge fixing operator and Bogomolny operator, respectively.

Proof of Proposition B.2. We construct Q^R as a conormal distribution on the double space \mathcal{Z}_ρ^2 , decomposing near $\mathcal{D} \cap \mathcal{B} \cap \Delta$ according to the splitting $\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_1$ as

$$\begin{aligned} Q^R &= \begin{pmatrix} \pi_L^*(\rho^{1/2}) \tilde{Q}_0 \pi_R^*(\rho^{-5/2}) & Q_{01} \\ Q_{10} & Q_1 \end{pmatrix} \\ \tilde{Q}_0 &\in \Psi_\rho^{-2, \mathcal{F}^0}(\mathcal{Z}; \mathfrak{p}_0), \quad Q_1 \in \Psi_\gamma^{-2, \mathcal{F}^1}(\mathcal{Z}; \mathfrak{p}_1), \\ Q_{ij} &\in \rho^\infty \Psi^{-2} \end{aligned} \tag{D.19}$$

for some index sets \mathcal{F}^i , $i = 1, 2$, determined below. $Q_1 \in \Psi_\gamma^{-2, \mathcal{F}^1}(\mathcal{Z}; \mathfrak{p}_1)$ is the pushforward to \mathcal{Z}_ρ^2 of an operator defined on the gluing double space \mathcal{Z}_γ^2 .

Working fiberwise, from Theorem C.11, the inverse of (C.31) for $\alpha = 0$ may be represented as a conormal distribution on X_b^2 with a decomposition (C.32). By the smoothness of $F|_X$ over \mathcal{I} , these inverses patch together smoothly as a family of distributions on $(\mathcal{X} \times_{\mathcal{I}} \mathcal{X})_b \cong \mathcal{X} \subset \mathcal{Z}_\rho^2$ over \mathcal{I} . Likewise, the fiberwise Fourier transforms of the symbolic inverses from Theorem C.13 form a smoothly varying family of conormal distributions on $\overline{\gamma T \mathcal{D}} \cong \mathcal{D}_{\text{sc}} \subset \mathcal{Z}_\rho^2$, and the inverses of (C.33) for $\alpha = \beta = 0$ form a smoothly varying family of distributions on $(\mathcal{D} \times_{\mathcal{I}} \mathcal{D})_b \cong \mathcal{D} \subset \mathcal{Z}_\rho^2$. Moreover, the leading order terms in the expansions of these inverses at $\mathcal{X} \cap \mathcal{D}$ are compatible; the scattering symbol of G_1 in (C.32) agrees with the inverse of the symbol in Theorem C.13 there, and after accounting for the various boundary defining factors, the indicial operators of \tilde{G}_0 in (C.32) and \tilde{G} in (C.34) agree.

Consequently, there exists a distribution Q on \mathcal{Z}_ρ^2 whose restriction to \mathcal{X} agrees with (C.32), and which decomposes as (D.19) where

$$\begin{aligned} \sigma_D(\tilde{Q}_0) &= \sigma_D(F_1)^{-1}, \quad \text{from (C.34),} \\ N_X(\tilde{Q}_0) &= \tilde{G}_0, \quad \text{near } X \cap D, \tilde{G}_0 \text{ from (C.32),} \\ I(N_X(\tilde{Q}_0), \lambda) &= I(N_D(\tilde{Q}_0), -\lambda), \end{aligned}$$

and $Q_1 \in \Psi_\gamma^{-2, \mathcal{F}^1}(\mathcal{Z}; \mathfrak{p}_0)$ satisfies

$$\begin{aligned} N_{D_{\text{sc}}}(Q_1) &= G_1, \quad G_1 \text{ from Theorem C.13} \\ N_{X_i}(Q_1) &= G_1, \quad G_1 \text{ from (C.32),} \\ \sigma_{\text{sc}}(N_{X_i}(Q_1)) &\cong \sigma_D(N_{D_{\text{sc}}}(Q_1)). \end{aligned}$$

Furthermore, by the usual iterative argument using principal symbols, we can arrange for the interior conormal singularity of Q^R to invert that of F to all orders.

The index sets \mathcal{F}^i , $i = 1, 2$ for Q^R satisfy

$$F_{\mathcal{D}}^0, F_{\mathcal{X}}^0, F_{\mathcal{B}}^0 \geq 0, \quad F_{\mathcal{D}}^0, F_{\mathcal{D}\mathcal{X}}^0, F_{\mathcal{B}\mathcal{Z}}^0, F_{\mathcal{Z}}^0 \geq \frac{1}{2}, \\ F_{\mathcal{D}_{\text{sc}}}^1, F_{\mathcal{B}_{\text{sc}}}^1, F_{\mathcal{X}}^1 \geq 0.$$

From Theorems D.7 and D.8, the error term $E' = I - Q^R F$ has (interior conormal) order $-\infty$, and admits a similar decomposition, with $E'_0 = \pi_L^*(\rho^{5/2})\tilde{E}'_0\pi_R^*(\rho^{-5/2})$, where $\tilde{E}'_0 \in \Psi_{\varrho}^{-\infty, \tilde{G}^0}(\mathcal{Z}; \mathfrak{p}_0)$ and $E'_1 \in \Psi_{\gamma}^{-\infty, G^1}(\mathcal{Z}; \mathfrak{p}_1)$ have the same estimates on their index sets as \tilde{Q}_0 and Q_1 , but with extra vanishing at the boundary faces meeting the diagonal over $\varepsilon = 0$ since they invert F there (for instance, $\tilde{G}_{\mathcal{D}}^0 = F_{\mathcal{D}} \setminus \min F_{\mathcal{D}}$, etc.). Thus

$$\tilde{G}_{\mathcal{D}}^0, \tilde{G}_{\mathcal{X}}^0, G_{\mathcal{D}_{\text{sc}}}^1, G_{\mathcal{X}}^1 > 0, \quad \tilde{G}_{\mathcal{B}}^0, G_{\mathcal{B}_{\text{sc}}}^1 \geq 0, \\ \tilde{G}_{\mathcal{D}\mathcal{X}}^0, \tilde{G}_{\mathcal{D}\mathcal{X}}^0, \tilde{G}_{\mathcal{B}\mathcal{Z}}^0, \tilde{G}_{\mathcal{Z}}^0 \geq \frac{1}{2},$$

Once we account for the left and right factors of ρ , it follows that $E'_0 = \pi_L^*(\rho^{5/2})\tilde{E}'_0\pi_R^*(\rho^{-5/2})$ is in $\Psi_{\varrho}^{-\infty, G^0}(\mathcal{Z}; \mathfrak{p}_0)$ where

$$G_{\mathcal{D}}^0, G_{\mathcal{X}}^0 > 0, \quad G_{\mathcal{B}}^0 \geq 0, \\ G_{\mathcal{D}\mathcal{X}}^0, G_{\mathcal{B}\mathcal{Z}}^0 \geq \frac{1}{2} + \frac{5}{2} = 3, \quad G_{\mathcal{D}}^0, G_{\mathcal{Z}}^0 \geq \frac{1}{2} - \frac{5}{2} = -2.$$

Now, on \mathcal{Z}_{ϱ}^2 (respectively \mathcal{Z}_{γ}^2), ε is a product of defining functions

$$\varepsilon = \rho_{\mathcal{D}}\rho_{\mathcal{X}}\rho_{\mathcal{D}\mathcal{X}}\rho_{\mathcal{D}\mathcal{X}}, \quad (\text{resp. } \varepsilon = \rho_{\mathcal{D}}\rho_{\mathcal{X}}\rho_{\mathcal{D}\mathcal{X}}\rho_{\mathcal{D}\mathcal{X}}\rho_{\mathcal{D}_{\text{sc}}}).$$

Taking

$$\delta < \min(G_{\mathcal{D}}^0, G_{\mathcal{X}}^0, G_{\mathcal{D}_{\text{sc}}}^1, G_{\mathcal{X}}^1, \frac{1}{2}),$$

it follows that we can write $E' = \varepsilon^{\delta} E^R$, where $E^R = \begin{pmatrix} E_0^R & E_{01}^R \\ E_{10}^R & E_1^R \end{pmatrix}$ satisfies

$E_0^R \in \Psi_{\varrho}^{-\infty, T^0}(\mathcal{Z}; \mathfrak{p}_0)$, $E_1^R \in \Psi_{\gamma}^{-\infty, T^1}(\mathcal{Z}; \mathfrak{p}_1)$, and

$$I_{\mathcal{D}}^0, I_{\mathcal{X}}^0, I_{\mathcal{D}_{\text{sc}}}^1, I_{\mathcal{X}}^1 > 0, \quad I_{\mathcal{B}}^0, I_{\mathcal{B}_{\text{sc}}}^1 \geq 0, \\ I_{\mathcal{D}\mathcal{X}}^0, I_{\mathcal{B}\mathcal{Z}}^0 > \frac{5}{2}, \quad I_{\mathcal{D}}^0, I_{\mathcal{Z}}^0 > -\frac{5}{2}.$$

By Theorems D.11 and D.12, it follows that

$$E_0^R : C^{\infty}(\mathcal{I}; \rho^1 H_{\mu, \varrho, \gamma}^{k, 0, 0}(Z; \mathfrak{p}_0)) \longrightarrow C^{\infty}(\mathcal{I}; \rho^1 H_{\mu, \varrho, \gamma}^{k, 0, 0}(Z; \mathfrak{p}_0)) \\ E_1^R : C^{\infty}(\mathcal{I}; \rho^{\beta} H_{\mu, \varrho, \gamma}^{k, 0, 0}(Z; \mathfrak{p}_1)) \longrightarrow C^{\infty}(\mathcal{I}; \rho^{\beta} H_{\mu, \varrho, \gamma}^{k, 0, 0}(Z; \mathfrak{p}_1)) \\ E_{01}^R : C^{\infty}(\mathcal{I}; \rho^{\beta} H_{\mu, \varrho, \gamma}^{k, 0, 0}(Z; \mathfrak{p}_1)) \longrightarrow C^{\infty}(\mathcal{I}; \rho^1 H_{\mu, \varrho, \gamma}^{k, 0, 0}(Z; \mathfrak{p}_0)) \\ E_{10}^R : C^{\infty}(\mathcal{I}; \rho^{\beta} H_{\mu, \varrho, \gamma}^{k, 0, 0}(Z; \mathfrak{p}_0)) \longrightarrow C^{\infty}(\mathcal{I}; \rho^1 H_{\mu, \varrho, \gamma}^{k, 0, 0}(Z; \mathfrak{p}_1))$$

are bounded (where defined), and hence E^R is a smooth bundle map of $\mathcal{H}^{k, 0, \beta}(Z; \mathfrak{p})$ over \mathcal{I} as claimed.

The construction of Q^L and estimates on E^L follow similarly. \square

Proof of Proposition 5.12. The construction is similar to the one in the previous proof so we shall be somewhat brief. We define R on \mathcal{Z}_ρ^2 , such that, with respect to the splitting $\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_1$ in a neighborhood of $\mathcal{D} \cap \mathcal{B} \cap \Delta$,

$$R = \begin{pmatrix} \pi_L^* \rho \tilde{R}_0 \pi_R^* \rho^{-2} & R_{01} \\ R_{10} & R_1 \end{pmatrix}$$

where $\rho := \rho_{\mathcal{D}} \rho_{\mathcal{B}}$, and $\tilde{R}_0 \in \Psi_\rho^{-1, \tilde{\mathcal{F}}^0}(\mathcal{Z}; \gamma\Lambda^* \otimes \mathfrak{p}_0)$, $R_1 \in \Psi_\gamma^{-1, \mathcal{F}^1}(\mathcal{Z}; \gamma\Lambda^* \otimes \mathfrak{p}_1)$, and $R_{ij} \in \rho^\infty \Psi^{-\infty}(\mathcal{Z}; \gamma\Lambda^* \otimes \mathfrak{p}_j, \gamma\Lambda^* \otimes \mathfrak{p}_i)$, where fiberwise over \mathcal{I} ,

$$\begin{aligned} N_D(\tilde{R}_0) &= \tilde{G}, \quad \tilde{G} \text{ from (C.28),} \\ N_X(\tilde{R}_0) &= \tilde{G}_0, \quad \text{near } X \cap D, \tilde{G}_0 \text{ from (C.19),} \\ I(N_X(\tilde{R}_0), \lambda) &= I(N_D(\tilde{R}_0), -\lambda), \end{aligned}$$

and $R_1 \in \Psi_\gamma^{-1, \mathcal{F}^1}(\mathcal{Z}; \mathfrak{p}_1)$ satisfies

$$\begin{aligned} \sigma_D(R_1) &= \sigma_D(L_1 + \Phi)^{-1}, \quad \text{from Proposition 5.6} \\ N_{X_i}(R_1) &= G_1, \quad G_1 \text{ from (C.19),} \\ \sigma_{\text{sc}}(N_{X_i}(R_1)) &\cong \sigma_{\mathcal{D}}(N_{D_{\text{sc}}}(R_1)). \end{aligned}$$

That such a R exists follows from compatibility of the inverses for L_{X_i} and L_D , which are smoothly parameterized over \mathcal{I} , and the fact that such distributions may be extended smoothly off the relevant boundary faces of \mathcal{Z}_ρ^2 and \mathcal{Z}_γ^2 . The index sets $\tilde{\mathcal{F}}^0$ and \mathcal{F}^1 satisfy

$$\begin{aligned} F_{\mathcal{D}}^0, F_{\mathcal{X}}^0, F_{\mathcal{B}}^0 &\geq 0, \quad F_{\mathcal{D}}^0, F_{\mathcal{X}}^0, F_{\mathcal{B}}^0 \geq 1 \\ F_{\mathcal{D}_{\text{sc}}}^1, F_{\mathcal{B}_{\text{sc}}}^1, F_{\mathcal{X}}^1 &\geq 0. \end{aligned}$$

We may further suppose that the interior conormal singularity of R inverts that of L to all orders.

It follows from Theorems D.7 and D.8 that

$$LR = I - E', \quad E' = \begin{pmatrix} E'_0 & E'_{01} \\ E'_{10} & E'_1 \end{pmatrix}$$

where $E'_0 = \pi_L^* \rho^2 \tilde{E}'_0 \pi_R^* \rho^{-2} \in \Psi_\rho^{-\infty, \mathcal{G}^0}(\mathcal{Z}; \gamma\Lambda^* \otimes \mathfrak{p})$, $E'_1 \in \Psi_\gamma^{-\infty, \mathcal{G}^1}(\mathcal{Z}; \gamma\Lambda^* \otimes \mathfrak{p})$ and $E'_{ij} \in \rho^\infty \Psi^{-\infty}(\mathcal{Z}; \gamma\Lambda^* \otimes \mathfrak{p}_j, \gamma\Lambda^* \otimes \mathfrak{p}_i)$. Here

$$\begin{aligned} G_{\mathcal{D}}^0, G_{\mathcal{X}}^0, G_{\mathcal{D}_{\text{sc}}}^1, G_{\mathcal{X}}^1 &> 0, \quad G_{\mathcal{B}}^0, G_{\mathcal{B}_{\text{sc}}}^1 \geq 0, \\ G_{\mathcal{D}}^0, G_{\mathcal{B}}^0 &\geq 1 + 2 = 3, \quad G_{\mathcal{D}}^0, G_{\mathcal{B}}^0 \geq 1 - 2 = -1. \end{aligned}$$

If we choose

$$\delta < \min(G_{\mathcal{D}}^0, G_{\mathcal{X}}^0, G_{\mathcal{D}_{\text{sc}}}^1, G_{\mathcal{X}}^1, \frac{1}{2}),$$

Then it follows that $E' =: \varepsilon^\delta E$, where $E = \begin{pmatrix} E_0 & E_{01} \\ E_{10} & E_1 \end{pmatrix}$ has index sets

$$\begin{aligned} E_0 &\in \Psi_\varrho^{-\infty, \mathcal{I}^0}, & E_1 &\in \Psi_\gamma^{-\infty, \mathcal{I}^1}, \\ I_{\mathcal{D}}^0, I_{\mathcal{X}}^0, I_{\mathcal{D}_{\text{sc}}}^1, I_{\mathcal{X}}^1 &> 0, & I_{\mathcal{B}}^0, I_{\mathcal{B}_{\text{sc}}}^1 &\geq 0, \\ I_{\mathcal{D}}^0, I_{\mathcal{B}}^0 &> \frac{5}{2}, & I_{\mathcal{X}}^0, I_{\mathcal{B}}^0 &> -\frac{3}{2} > -\frac{5}{2}. \end{aligned}$$

Boundedness of E as an operator (5.19) follows from Theorems D.7 and D.8. \square

REFERENCES

- [AH88] M.F. Atiyah and N.J. Hitchin, *The geometry and dynamics of magnetic monopoles*, Princeton University Press Princeton, NJ, 1988. 3, 13, 15, 20, 21
- [Bie] R. Bielawski, *Asymptotics of monopole metrics revisited*, In preparation. 6, 9
- [Bie98] Roger Bielawski, *Monopoles and the Gibbons-Manton metric*, Comm. Math. Phys. **194** (1998), no. 2, 297–321. MR 1627653 (2000a:53038) 4
- [Bie08] R. Bielawski, *Monopoles and clusters*, Communications in Mathematical Physics **284** (2008), no. 3, 675–712. 4
- [Don84] S.K. Donaldson, *Nahm's equations and the classification of monopoles*, Communications in Mathematical Physics **96** (1984), no. 3, 387–407. 5
- [Fos14] Lorenzo Foscolo, *A gluing construction for periodic monopoles*, arXiv:1411.6951, 2014. 5
- [GM95] GW Gibbons and NS Manton, *The moduli space metric for well-separated bps monopoles*, Physics Letters B **356** (1995), no. 1, 32–38. 4, 6
- [HKLR87] Nigel J Hitchin, Anders Karlhede, Ulf Lindström, and M Roček, *Hyperkähler metrics and supersymmetry*, Communications in Mathematical Physics **108** (1987), no. 4, 535–589. 3
- [Joy15] D. Joyce, *A generalization of manifolds with corners*, arXiv:1501.00401, 2015. 90
- [JT80] A. Jaffe and C. Taubes, *Vortices and monopoles: structure of static gauge theories*, Birkhauser, Boston, 1980. 5, 10, 13, 14, 15
- [KM15] C. Kottke and R.B. Melrose, *Generalized blow up of corners and fiber products*, Transactions of the AMS **367** (2015), no. 1, 651–705. 89, 90, 94, 98, 99
- [Kot15a] C. Kottke, *A Callias-type index theorem with degenerate potentials*, Communications in PDE **40** (2015), no. 2, 219–264. 73
- [Kot15b] ———, *Blow-up in manifolds with generalized corners*, arXiv:1509.03874, 2015. 89, 90, 98
- [Kot15c] ———, *Dimension of monopoles on asymptotically conic 3-manifolds*, Bulletin of the LMS **47** (2015), no. 5, 818–834. 19, 76, 78, 79
- [Mel92] R.B. Melrose, *Calculus of conormal distributions on manifolds with corners*, International Mathematics Research Notices **1992** (1992), no. 3, 51–61. 8, 28, 32, 40, 102, 105
- [Mel93] ———, *The Atiyah-Patodi-Singer index theorem*, Research Notes in Mathematics, vol. 4, AK Peters, Ltd, Wellesley MA, 1993. 13, 27, 79, 82
- [Mel94] R. Melrose, *Spectral and scattering theory for the Laplacian on asymptotically Euclidian spaces*, Spectral and scattering theory: proceedings of the Taniguchi international workshop, CRC Press, 1994, pp. 85–130. 12, 85
- [MM98] Rafe Mazzeo and Richard B. Melrose, *Pseudodifferential operators on manifolds with fibred boundaries*, Asian J. Math. **2** (1998), no. 4, 833–866, Mikio Sato: a great Japanese mathematician of the twentieth century. MR 1734130 (2000m:58046) 54

- [Oli14] Gonalo Oliveira, *Monopoles on 3 dimensional AC manifolds*, arXiv:1412.2252, 2014. 5, 23
- [Pol74] A. M. Polyakov, *Particle spectrum in quantum field theory*, JETP Lett. **20** (1974), 194–195. 15
- [Tau83] C.H. Taubes, *Stability in Yang-Mills theories*, Communications in Mathematical Physics **91** (1983), no. 2, 235–263. 3, 19
- [Tau85] ———, *Min-max theory for the Yang-Mills-Higgs equations*, Communications in Mathematical Physics **97** (1985), no. 4, 473–540. 3, 4
- [tH74] G. 't Hooft, *Magnetic monopoles in unified gauge theories*, Nuclear Phys. **B79** (1974), 276–284. MR 0413809 (54 #1923) 15

NORTHEASTERN UNIVERSITY, DEPARTMENT OF MATHEMATICS
E-mail address: `c.kottke@neu.edu`

UNIVERSITY COLLEGE LONDON, DEPARTMENT OF MATHEMATICS
E-mail address: `michael.singer@ucl.ac.uk`